Orthogonal Polynomials appearing in SIE grid representations

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Abstract

We show in this article how orthogonal polynomials appear in certain representations of grid shaped quivers. After a short introduction into the general notion of quivers and their representations by linear operators we define the notion of an SIE quiver representation: All intrinsic endomorphisms arising from circuits in the quiver act as scalar multipliers. We then present several lemmas that ensure this SIE property of a quiver representation. Ladder commutator conditions and certain diagram commutativity "up to scalar multiples" play a central role. The theory will then be applied to three examples. Extensive calculations shows how Associated Laguerre, Legendre–Gegenbauer polynomials and binomial distributions fit into the framework of grid shaped SIE quivers. One can see, that this algebraic point of view is foundational for orthogonal polynomials and special functions.

Keywords Orthogonal polynomials, Quiver representations, Ladders, Weyl algebra, Differential and Difference Operators

MSC 33C45 16G20

1 Introduction

We show in this article how orthogonal polynomials appear in certain representations of grid shaped quivers. After a short introduction into the general notion of quivers and their representations by linear operators between vector spaces we define the notion of an SIE quiver representation: All intrinsic endomorphisms arising from circuits in the quiver act as scalar multiples. We then present several theorems that ensure this SIE property of a quiver representation. Ladder commutator conditions and certain diagram commutativity "up to scalar multiples" play a central role. The theory will then be applied to three examples. Extensive calculations show how Laguerre, Legendre–Gegenbauer polynomials and the discrete binomial distribution fit into the framework of grid shaped SIE quivers. This point of view onto orthogonal polynomials seems not to be known, see [2], [8], [7], e.g. Note that we study operator algebraic aspects of orthogonal polynomials, we do not pursue orthogonality in the literal sense or functional analytic or numerical approaches.

An enhancement of the theory including further examples of orthogonal polynomials, q–orthogonal polynomials connected by operators of the Weyl algebra, h–Weyl algebra, q–Weyl algebra or (q, h)–Weyl algebra, see [4], is not included here.

2 Grids

In this article we will study so called grids of operators. Let $\mathcal{A}$ be a fixed associative unital algebra over $\mathbb{C}$ with a representation on a vector space V. A grid is a directed graph with vertices $\bullet_{nm}$, indexed by pairs $(n, m) \in \mathbb{Z}\times\mathbb{Z}$. Between two neighbouring vertices there is a pair of arrows, one in each direction. There are horizontal, vertical and diagonal arrows. The local

situation at the vertex $\bullet_{nm}$ is shown in the diagram

$$
\begin{tikzcd}
\bullet_{n-1,m+1} \arrow[r, shift left, "a^+_{n,m+1}"] \arrow[d, shift left, "b^-_{n-1,m+1}"] \arrow[dr, shift left, "q^-_{n-1,m+1}"] & \bullet_{n,m+1} \arrow[l, shift left, "a^-_{n,m+1}"] \arrow[r, shift left, "a^+_{n+1,m+1}"] \arrow[d, shift left, "b^-_{n,m+1}"] & \bullet_{n+1,m+1} \arrow[l, shift left, "a^-_{n+1,m+1}"] \arrow[d, shift left, "b^-_{n+1,m+1}"] \arrow[dl, shift left, "p^-_{n+1,m+1}"] \\
\bullet_{n-1,m} \arrow[u, shift left, "b^+_{n-1,m+1}"] \arrow[r, shift left, "a^+_{n,m}"] \arrow[d, shift left, "b^-_{n-1,m}"] & \bullet_{n,m} \arrow[ul, shift left, "q^+_{n-1,m+1}"] \arrow[u, shift left, "b^+_{n,m+1}"] \arrow[ur, shift left, "p^+_{n+1,m+1}"] \arrow[l, shift left, "a^-_{n,m}"] \arrow[r, shift left, "a^+_{n+1,m}"] \arrow[d, shift left, "b^-_{n,m}"] \arrow[dl, shift left, "p^-_{n,m}"] \arrow[dr, shift left, "q^-_{n,m}"] & \bullet_{n+1,m} \arrow[l, shift left, "a^-_{n+1,m}"] \arrow[u, shift left, "b^+_{n+1,m+1}"] \arrow[d, shift left, "b^-_{n+1,m}"] \\
\bullet_{n-1,m-1} \arrow[u, shift left, "b^+_{n-1,m}"] \arrow[r, shift left, "a^+_{n,m-1}"] \arrow[ur, shift left, "p^+_{n,m}"] & \bullet_{n,m-1} \arrow[l, shift left, "a^-_{n,m-1}"] \arrow[u, shift left, "b^+_{n,m}"] \arrow[r, shift left, "a^+_{n+1,m-1}"] & \bullet_{n+1,m-1} \arrow[l, shift left, "a^-_{n+1,m-1}"] \arrow[u, shift left, "b^+_{n+1,m}"] \arrow[ul, shift left, "q^+_{n,m}"]
\end{tikzcd}
\tag{1}
$$

One can see that we have adopted the following general convention for indices of arrows: An arrow with superscript $+$ has the index of its target vertex. An arrow with superscript $-$ has the index of its source vertex.

3 Quivers

We consider a grid as a special quiver. Instead of a rigoros definition we describe the notion of a quiver as follows: A *quiver* is a directed graph G. It consists of a set $\mathbf{P} = \{\bullet_i | i \in I\}$ of points and a set $\mathbf{A} = \{a_j | j \in J\}$, of arrows connecting the vertices. More accurately, to each arrow a one can assign a source vertex $s(a)$ and a target vertex $t(a)$ in $\mathbf{P}$. The quivers which we are dealing with are infinite, they will have circuits, multiple arrows are allowed.

A path in G is a sequence $(a_m\, a_{m-1} \cdots a_2\, a_1)$ of consecutive arrows in $\mathbf{A}$, that is

$$t(a_i) = s(a_{i+1}) \qquad \text{for all } i = 1, \ldots, m-1.$$

We can extend the source and target maps to the set of paths by

$$s(a_m \cdots a_1) = s(a_1), \qquad t(a_m \cdots a_1) = t(a_m).$$

If $a = (a_m \cdots a_1), \widetilde{a} = (\widetilde{a}_{\widetilde{m}} \cdots \widetilde{a}_1)$ are two paths with $t(a) = s(\widetilde{a})$ we define their product by concatenation or arrows

$$(\widetilde{a}_{\widetilde{m}} \cdots \widetilde{a}_1) \cdot (a_m \cdots a_1) = (\widetilde{a}_{\widetilde{m}} \cdots \widetilde{a}_1\, a_m \cdots a_1).$$

In this article we will study grids or quivers that only have double arrows, as shown in the above diagram. In mathematical terms this means that that we can assign to each arrow a a unique arrow a^{op} between the same vertices pointing in the opposite direction, that is

$$t(a^{\mathrm{op}}) = s(a), \qquad s(a^{\mathrm{op}}) = t(a).$$

Then for every path $a = (a_m \cdots a_1)$ there is an *opposite* path,

$$(a_m \cdots a_1)^{\mathrm{op}} = (a_1^{\mathrm{op}} \cdots a_m^{\mathrm{op}}).$$

A path a is called a *circuit* if $s(a) = t(a)$. A circuit a is called *narrow*, if there is a path b, such that $a = b^{\mathrm{op}} b$. This means that the circuit the path b and then returns along the same path in opposite direction. Circuits that are not narrow will be called *wide*. A narrow circuit a is called *short*, if there is an arrow b, such that $a = b^{\mathrm{op}} b$.

4 Representations of Quivers

Like every directed graph, a quiver or grid can be considered as a category with the vertices as objects and the arrows as morphisms. A *representation* ϱ of this quiver is just a functor from this category to the category of (complex) vector spaces. In other words it is an assignment

$$\varrho : \begin{cases} \bullet_i & \mapsto & V_i \\ a_j & \mapsto & A_j \end{cases}$$

of vertices $\bullet_i$ to vector spaces V_i and of arrows a_j to linear operators A_j acting between these vector spaces in the obvious way:

$$A_j : V_{s(a_j)} \to V_{t(a_j)}.$$

So, a representation of a quiver is just a diagram of vector spaces and operators such that the quiver carries the information about the diagram structure. This situation is shown in the following diagram, the grid shape serves

as an example again.

$$
\begin{tikzcd}[column sep=large, row sep=large]
V_{n-1,m+1} \arrow[r, shift left, "A^+_{n,m+1}"] \arrow[d, shift left, "B^-_{n-1,m+1}"] \arrow[dr, shift left, "Q^-_{n-1,m+1}"] & V_{n,m+1} \arrow[l, shift left, "A^-_{n,m+1}"] \arrow[r, shift left, "A^+_{n+1,m+1}"] \arrow[d, shift left, "B^-_{n,m+1}"] & V_{n+1,m+1} \arrow[l, shift left, "A^-_{n+1,m+1}"] \arrow[d, shift left, "B^-_{n+1,m+1}"] \arrow[dl, shift left, "P^-_{n+1,m+1}"] \\
V_{n-1,m} \arrow[r, shift left, "A^+_{n,m}"] \arrow[u, shift left, "B^+_{n-1,m+1}"] \arrow[d, shift left, "B^-_{n-1,m}"] & V_{n,m} \arrow[l, shift left, "A^-_{n,m}"] \arrow[r, shift left, "A^+_{n+1,m}"] \arrow[u, shift left, "B^+_{n,m+1}"] \arrow[d, shift left, "B^-_{n,m}"] \arrow[ul, shift left, "Q^+_{n-1,m+1}"] \arrow[ur, shift left, "P^+_{n+1,m+1}"] \arrow[dl, shift left, "P^-_{n,m}"] \arrow[dr, shift left, "Q^-_{n,m}"] & V_{n+1,m} \arrow[l, shift left, "A^-_{n+1,m}"] \arrow[u, shift left, "B^+_{n+1,m+1}"] \arrow[d, shift left, "B^-_{n+1,m}"] \\
V_{n-1,m-1} \arrow[r, shift left, "A^+_{n,m-1}"] \arrow[u, shift left, "B^+_{n-1,m}"] \arrow[ur, shift left, "P^+_{n,m}"] & V_{n,m-1} \arrow[l, shift left, "A^-_{n,m-1}"] \arrow[r, shift left, "A^+_{n+1,m-1}"] \arrow[u, shift left, "B^+_{n,m}"] & V_{n+1,m-1} \arrow[l, shift left, "A^-_{n+1,m-1}"] \arrow[u, shift left, "B^+_{n+1,m}"] \arrow[ul, shift left, "Q^+_{n,m}"]
\end{tikzcd}
\tag{2}
$$

A *homomorphism* of two representations (V_i, A_j) and (W_i, B_j) is just a collection of linear operators $(\varphi_i : V_i \to W_i)_{i\in I}$ that commute with the arrow operators A_j in the right way: For any arrow a_j in the quiver we have

$$B_j\,\varphi_{s(j)} \;=\; \varphi_{t(j)}\,A_j \quad : \quad V_{s(j)} \to W_{t(j)}.$$

To each path $a = (a_m \cdots a_1)$ we can assign the operator product $A = A_m \cdots A_1$. The path product corresponds to the operator product. An operator assigned to a circuit is a vector space endomorphism. We will call it a *circuit endomorphism*.

Now we consider a fixed representation ϱ of a quiver. We say that a circuit endomorphism $A_j : V_{s(j)} \to V_{s(j)}$ *acts as a scalar* or it is a ***S****calar* ***I****ntrinsic* ***E****ndomorphism (SIE)*, if it is a scalar multiple of the identity mapping on $V_{s(j)}$

$$A_j \;=\; \alpha \cdot \mathrm{id}_{V_{s(j)}} \;=\; \alpha \qquad \text{for some } \alpha \in \mathbb{C}.$$

Here and from now on we identify a scalar multiple of the identity operator with the scalar itself.

The quiver representation ϱ will be called an *SIE representation*, if all circuit endomorphisms arising in it are SIE. The following lemmas collect some simple linear algebraic observations about SIEs.

Lemma 1 (Narrow circuit endomorphisms) *Let a be a path in the quiver and $A = \varrho(a), A^{\mathrm{op}} = \varrho(a^{\mathrm{op}})$ the corresponding operators in both directions*

$$V_{s(a)} \quad \underset{A^{\mathrm{op}}}{\overset{A}{\rightleftarrows}} \quad V_{t(a)}.$$

(i) If the two narrow circuit endomorphisms $A^{\mathrm{op}}A$ and AA^{op} act as scalars

$$A^{\mathrm{op}}A \;=\; \alpha, \qquad AA^{\mathrm{op}} \;=\; \widetilde{\alpha},$$

then these scalars are equal: $\alpha = \widetilde{\alpha}$.

The example $\mathbb{R} \;\underset{\mathrm{pr}_1}{\overset{\mathrm{id}\times 0}{\rightleftarrows}}\; \mathbb{R}^2$ shows that the two conditions do not imply each other.

(iii) Let $\alpha \in \mathbb{C}\setminus\{0\}$ be a nonzero number. The following statements about the corresponding circuit endomorphisms are equivalent

$A^{\mathrm{op}}A = \alpha\ \mathrm{id}_V$ *and A surjective.*

$A^{\mathrm{op}}A = \alpha\ \mathrm{id}_V$ *and A^{op} injective.*

$AA^{\mathrm{op}} = \alpha\ \mathrm{id}_W$ *and A^{op} surjective.*

$AA^{\mathrm{op}} = \alpha\ \mathrm{id}_W$ *and A injective.*

$A^{\mathrm{op}}A = \alpha\ \mathrm{id}_V$ *and $AA^{\mathrm{op}} = \alpha\ \mathrm{id}_W$. V and W are isomorphic.*

(iv) If all short circuit endomorphisms act as scalars then all circuit endomorphisms act as scalars.

Proof The proof of (iii) is simply based on the fact that $BC = \alpha\cdot\mathrm{id}$ implies B injective and C surjective.

We are going to show (iv): For a path $a = (a_m \cdots a_1)$ of arrows and its operator counterpart $A = A_m \cdots A_1$ we have the identity

$$A^{\mathrm{op}}A \;=\; A_1^{\mathrm{op}} \cdots \underbrace{A_{m-1}^{\mathrm{op}}\,\underbrace{A_m^{\mathrm{op}}A_m}\,A_{m-1}} \cdots A_1 \;=\; \alpha_m \cdots \alpha_m$$

Inductively — starting with the innermost circuit operator $A_m^{\mathrm{op}}A_m$ — all the circuits act as scalars and commute with the outer operators. ⧫

Lemma 2 (Wide circuit endomorphisms) *Consider the following part of a representation (U_{nm}) of a grid.*

$$\begin{tikzcd} V_{n,m+1} \arrow[r, shift left, "A^+_{n+1,m+1}"] \arrow[d, shift left, "B^-_{n,m+1}"] & V_{n+1,m+1} \arrow[l, shift left, "A^-_{n+1,m+1}"] \arrow[d, shift left, "B^-_{n+1,m+1}"] \\ V_{n,m} \arrow[r, shift left, "A^+_{n+1,m}"] \arrow[u, shift left, "B^+_{n,m+1}"] & V_{n+1,m} \arrow[l, shift left, "A^-_{n+1,m}"] \arrow[u, shift left, "B^+_{n+1,m+1}"] \end{tikzcd} \tag{3}$$

Assume that the eight narrow circuit endomorphisms act as scalars as follows:

$$\begin{aligned} A^-_{n+1,m}A^+_{n+1,m} &= \alpha_{n+1,m} & A^+_{n+1,m}A^-_{n+1,m} &= \alpha_{n+1,m} \\ B^-_{n+1,m+1}B^+_{n+1,m+1} &= \beta_{n+1,m+1} & B^+_{n+1,m+1}B^-_{n+1,m+1} &= \beta_{n+1,m+1} \\ A^+_{n+1,m+1}A^-_{n+1,m+1} &= \alpha_{n+1,m+1} & A^-_{n+1,m+1}A^+_{n+1,m+1} &= \alpha_{n+1,m+1} \\ B^+_{n,m+1}B^-_{n,m+1} &= \beta_{n,m+1} & B^-_{n,m+1}B^+_{n,m+1} &= \beta_{n,m+1} \end{aligned} \tag{4}$$

Let γ_{nm}, δ_{nm} be two more numbers such that their product equals the product of the four numbers

$$\beta_{n,m+1} \cdot \alpha_{n+1,m+1} \cdot \beta_{n+1,m+1} \cdot \alpha_{n+1,m} \quad = \quad \gamma_{nm} \cdot \delta_{nm}. \tag{5}$$

Then the following sixteen equations are equivalent.

$$
\begin{array}{llrcl}
\circlearrowleft: V_{n,m} & \to V_{n,m} & B^-_{n,m+1} \cdot A^-_{n+1,m+1} \cdot B^+_{n+1,m+1} \cdot A^+_{n+1,m} & = & \gamma_{nm} \\
\circlearrowleft: V_{n+1,m} & \to V_{n+1,m} & A^+_{n+1,m} \cdot B^-_{n,m+1} \cdot A^-_{n+1,m+1} \cdot B^+_{n+1,m+1} & = & \gamma_{nm} \\
\circlearrowleft: V_{n+1,m+1} & \to V_{n+1,m+1} & B^+_{n+1,m+1} \cdot A^+_{n+1,m} \cdot B^-_{n,m+1} \cdot A^-_{n+1,m+1} & = & \gamma_{nm} \\
\circlearrowleft: V_{n,m+1} & \to V_{n,m+1} & A^-_{n+1,m+1} \cdot B^+_{n+1,m+1} \cdot A^+_{n+1,m} \cdot B^-_{n,m+1} & = & \gamma_{nm} \\
\circlearrowright: V_{n,m} & \to V_{n,m} & A^-_{n+1,m} \cdot B^-_{n+1,m+1} \cdot A^+_{n+1,m+1} \cdot B^+_{n,m+1} & = & \delta_{nm} \\
\circlearrowright: V_{n,m+1} & \to V_{n,m+1} & B^+_{n,m+1} \cdot A^-_{n+1,m} \cdot B^-_{n+1,m+1} \cdot A^+_{n+1,m+1} & = & \delta_{nm} \\
\circlearrowright: V_{n+1,m+1} & \to V_{n+1,m+1} & A^+_{n+1,m+1} \cdot B^+_{n,m+1} \cdot A^-_{n+1,m} \cdot B^-_{n+1,m+1} & = & \delta_{nm} \\
\circlearrowright: V_{n+1,m} & \to V_{n+1,m} & B^-_{n+1,m+1} \cdot A^+_{n+1,m+1} \cdot B^+_{n,m+1} \cdot A^-_{n+1,m} & = & \delta_{nm}
\end{array}
$$

$$
\begin{array}{lllcll}
\nearrow: V_{n,m} & \to V_{n+1,m+1} & A^+_{n+1,m+1} \cdot B^+_{n,m+1} & = & \frac{\alpha_{n+1,m+1} \cdot \beta_{n,m+1}}{\gamma_{nm}} & B^+_{n+1,m+1} \cdot A^+_{n+1,m} \\
\nwarrow: V_{n+1,m} & \to V_{n,m+1} & B^+_{n,m+1} \cdot A^-_{n+1,m} & = & \frac{\alpha_{n+1,m} \cdot \beta_{n,m+1}}{\gamma_{nm}} & A^-_{n+1,m+1} \cdot B^+_{n+1,m+1} \\
\swarrow: V_{n+1,m+1} & \to V_{n,m} & A^-_{n+1,m} \cdot B^-_{n+1,m+1} & = & \frac{\alpha_{n+1,m} \cdot \beta_{n+1,m+1}}{\gamma_{nm}} & B^-_{n,m+1} \cdot A^-_{n+1,m+1} \\
\searrow: V_{n,m+1} & \to V_{n+1,m} & B^-_{n+1,m+1} \cdot A^+_{n+1,m+1} & = & \frac{\alpha_{n+1,m+1} \cdot \beta_{n+1,m+1}}{\gamma_{nm}} & A^+_{n+1,m} \cdot B^-_{n,m+1} \\
\nearrow: V_{n,m} & \to V_{n+1,m+1} & B^+_{n+1,m+1} \cdot A^+_{n+1,m} & = & \frac{\alpha_{n+1,m} \cdot \beta_{n+1,m+1}}{\delta_{nm}} & A^+_{n+1,m+1} \cdot B^+_{n,m+1} \\
\nwarrow: V_{n+1,m} & \to V_{n,m+1} & A^-_{n+1,m+1} \cdot B^+_{n+1,m+1} & = & \frac{\alpha_{n+1,m+1} \cdot \beta_{n+1,m+1}}{\delta_{nm}} & B^+_{n,m+1} \cdot A^-_{n+1,m} \\
\swarrow: V_{n+1,m+1} & \to V_{n,m} & B^-_{n,m+1} \cdot A^-_{n+1,m+1} & = & \frac{\alpha_{n+1,m+1} \cdot \beta_{n,m+1}}{\delta_{nm}} & A^-_{n+1,m} \cdot B^-_{n+1,m+1} \\
\searrow: V_{n,m+1} & \to V_{n+1,m} & A^+_{n+1,m} \cdot B^-_{n,m+1} & = & \frac{\alpha_{n+1,m} \cdot \beta_{n,m+1}}{\delta_{nm}} & B^-_{n+1,m+1} \cdot A^+_{n+1,m+1}
\end{array}
$$

The first four equations state that the wide circuit endomorphisms along the square in counterclockwise direction act as the same scalar. The next four equations contain the same statement for the clockwise direction. The other eight equations describe the "up to scalar commutativity" of the diagram.

Proof Just multiply the first equation with $A^-_{n+1,m}$ from the right and with $A^+_{n+1,m}$ from the left. Then the second identity in (4) proves the second equation.

Multiply the first equation with the operator of the fifth line from the left. Then the left side of the identities in (4) and the identity (5) show the fifth equation.

Multiply the first equation with the left operator of the ninth line from the left. Then the fifth and seventh identity in (4) show the ninth equation.

The equivalence of the ninth and tenth equation follows directly from (5).

Cyclic commutation shows all the other implications. ♦

5 Ladders

We now consider a single (finite or infinite) path in the quiver together with its opposite path. A representation of a path, called a *ladder*, is illustrated by the diagram

$$\cdots \quad V_{n-1} \quad \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} \quad V_n \quad \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{6}$$

Ladders appear abundantly in quantum mechanics and quantum field theory (see [1]). In [3] they are used for studying Markov chains on trees. A basic study of this notion and many examples can be found in [6]. The following theorem from ladder theory is taken from [6], the setting here is a little bit different, mainly due to other index conventions.

For a given general ladder we define the narrow circuit operators

$$A_n^{\sqsubset} \;:=\; A_n^+ A_n^-, \qquad A_n^{\sqsupset} \;:=\; A_{n+1}^- A_{n+1}^+.$$

Lemma 3 (Scalar commutators) *Assume that there is a sequence* $(\alpha_n)_{n\in\mathbb{Z}}$ *of (real or complex) numbers, such that*

$$A_n^{\sqsupset} - A_n^{\sqsubset} \;=\; A_{n+1}^- A_{n+1}^+ - A_n^+ A_n^- \;=\; \alpha_{n+1} - \alpha_n \quad \textit{for all } n \in \mathbb{Z} \tag{7}$$

Then the ladder

$$\cdots \quad \mathcal{E}_{n-1} \quad \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} \quad \mathcal{E}_n \quad \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} \quad \mathcal{E}_{n+1} \quad \cdots \tag{8}$$

with eigenspaces

$$\mathcal{E}_n \;:=\; \mathrm{eig}(A_n^{\sqsubset}, \alpha_n) \;=\; \mathrm{eig}(A_n^{\sqsupset}, \alpha_{n+1})$$

is a well defined SIE subladder of (6).

Proof The identity (7) ensures that the two different expressions in the definition of $\mathcal{E}_n$ are equivalent. Now let $v \in \mathcal{E}_n$. Then

$$\begin{aligned} & (A_{n+1}^+ A_{n+1}^-) A_{n+1}^+ v \;=\; A_{n+1}^+ (A_{n+1}^- A_{n+1}^+) v \\ =\; & A_{n+1}^+ \alpha_{n+1} v \;=\; \alpha_{n+1} A_{n+1}^+ v, \end{aligned}$$

so $A_{n+1}^+ v \in \mathcal{E}_{n+1}$. On the other side we have

$$(A_n^- A_n^+) A_n^- v \;=\; A_n^- (A_n^+ A_n^-) v \;=\; A_n^- \alpha_n v \;=\; \alpha_n A_n^- v,$$

but this is $A_n^- v \in \mathcal{E}_{n-1}$. ♦

6 The Weyl algebra

The Weyl algebra is generated as an associative unital $\mathbb{C}$ algebra by the two operators D, X subject to the canonical commutator relation

$$[D, X] = 1.$$

The standard representation is by operators acting on $\mathcal{C}^\infty(\mathbb{R})$ via differentiation (D) and multiplication with the variable (X), respectively. If p is a polynomial over $\mathbb{C}$, then

$$[D, p(X)] = p'(X), \qquad [p(D), X] = p'(D).$$

In the Weyl algebra we additionally define the operators

$$\begin{aligned} C &:= D - 1 \\ \mathcal{D}_n &:= 1 + D + D^2 + \ldots + D^n \end{aligned}$$

Then

$$[C, D] = 0, \qquad [C, X] = 1, \qquad C\mathcal{D}_n = \mathcal{D}_n C \;=\; D^{n+1} - 1.$$

7 The Laguerre grid

In this section we assume that the operator X in the Weyl algebra is invertible. On the algebraic level this can be achieved by adjoining X^{-1} to the Weyl algebra. With respect to the standard representation we have to restrict the domain of the relevant functions, to $\mathbb{R}^+$, e.g.

Now, for $n \in \mathbb{N}_0, k \in \mathbb{Z}$ define the (second order differential) *Laguerre operator*

$$H_{nk} \quad := \quad H_{nk}^{\text{Lag}} \;:=\; XCD + (k+1)D + n \;=\; CXD + kD + n$$

and — according to some fixed representation of the Weyl algebra — the spaces

$$U_{nk} \;:=\; \ker H_{nk} \cap \ker D^{n+1}.$$

The diagram below shows locally the so-called *Laguerre grid*. The global Laguerre grid is defined for $n \in \mathbb{N}_0, k \in \mathbb{Z}$. Note that the second index

$k := m - n$ encodes the numbers of the NE/SW diagonals instead of the horizontal rows.

$$
\begin{tikzcd}[column sep=large, row sep=large]
U_{n-1,k+2} \arrow[r, shift left, "k+2+XC"] \arrow[d, shift left, "\mathcal{D}_{n-1}"] \arrow[dr, shift left, "(k+1)\mathcal{D}_{n-1}-X"'] & U_{n,k+1} \arrow[l, shift left, "-D"] \arrow[r, shift left, "k+1+XC"] \arrow[d, shift left, "\mathcal{D}_n"] & U_{n+1,k} \arrow[l, shift left, "-D"] \arrow[d, shift left, "\mathcal{D}_{n+1}"] \arrow[dl, shift left, "n+1-XD"] \\
U_{n-1,k+1} \arrow[u, shift left, "-C"] \arrow[r, shift left, "k+1+XC"] \arrow[d, shift left, "\mathcal{D}_{n-1}"] & U_{n,k} \arrow[ul, shift left, "DC"] \arrow[ur, shift left, "n+k+1+XC"] \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[r, shift left, "k+XC"] \arrow[d, shift left, "\mathcal{D}_n"] \arrow[dl, shift left, "n-XD"] \arrow[dr, shift left, "(k-1)\mathcal{D}_n-X"'] & U_{n+1,k-1} \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[d, shift left, "\mathcal{D}_{n+1}"] \\
U_{n-1,k} \arrow[u, shift left, "-C"] \arrow[ur, shift left, "n+k+XC"] \arrow[r, shift left, "k+XC"] & U_{n,k-1} \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[r, shift left, "k-1+XC"] & U_{n+1,k-2} \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[ul, shift left, "DC"]
\end{tikzcd}
\tag{9}
$$

Theorem 4 (Laguerre grid) *The Laguerre grid is a well defined SIE grid. The various circuit endomorphisms $U_{nk} \to U_{nk}$ act as scalars as follows*

$\rightleftharpoons$ *East*	$(-D)(k+XC)$	$= n+1$
$\rightleftharpoons$ *West*	$(-D)(k+1+XC)$	$= n$
$\rightleftharpoons$ *North*	$\mathcal{D}_n(-C)$	$= 1$
$\rightleftharpoons$ *South*	$(-C)\mathcal{D}_n$	$= 1$
$\rightleftharpoons$ *Northeast*	$(n+1-XD)(n+1+k+XC)$	$= (n+1)(n+k+1)$
$\rightleftharpoons$ *Southwest*	$[(n+k)+XC][n-XD]$	$= n(n+k)$
$\rightleftharpoons$ *Northwest*	$[(k+1)\mathcal{D}_{n-1}-X]DC$	$= n$
$\rightleftharpoons$ *Southeast*	$DC[(k-1)\mathcal{D}_{n-1}-X]$	$= n+1$
$\triangle$ *East-Northeast*	$(n+1-XD)(-C)(k+XC)$	$= (n+1)(n+k+1)$
$\triangle$ *Northeast-East*	$(-D)\mathcal{D}_{n+1}(n+k+1+XC)$	$= n+1$
$\triangle$ *Northeast-North*	$\mathcal{D}_n(-D)(n+k+1+XC)$	$= n+1$
$\triangle$ *North-Northeast*	$(n+1-XD)(k+1+XC)(-C)$	$= (n+1)(n+k+1)$
$\triangle$ *North-Northwest*	$[(k+1)\mathcal{D}_{n-1}-X](-D)(-C)$	$= n$
$\triangle$ *Northwest-North*	$\mathcal{D}_n(k+2+XC)DC$	$= n$
$\triangle$ *Northwest-West*	$(k+1+XC)\mathcal{D}_{n-1}DC$	$= n$
$\triangle$ *West-Northwest*	$[(k+1)\mathcal{D}_{n-1}-X](-C)(-D)$	$= n$
$\square$ $\circlearrowleft$	$\mathcal{D}_n\,(-D)\,(-C)\,(k+XC)$	$= n+1$
$\square$ $\circlearrowright$	$(-D)\,\mathcal{D}_{n+1}\,(k+1+XC)\,(-C)$	$= n+1$

The double arrow symbolizes the narrow circuit endomorphisms in the direction as stated. The triangle symbols $\triangle$ denote the various triangle circuit endomorphisms, the square symbols $\square$ denote the square circuit endomorphisms in clockwise and counterclockwise direction, respectively.

Proof (1) The horizontal narrow circuit endomorphisms in East and West direction are:

$$A_{nk}^{\sqsupset} = (-D)(k+XC) = -XCD-(k+1)D+1 = -H_{nk}+n+1$$

$$A_{nk}^{\sqsubset} = (k+1+XC)(-D) = -XCD-(k+1)D = -H_{nk}+n$$

When choosing the number sequence $\alpha_{nk} = n$, we find that the ladder commutator condition (7) in Lemma 3 is fulfilled,

$$A^{\sqsupset}_{nk} - A^{\sqsubset}_{nk} \quad = \quad 1 \ = \ \alpha_{n+1,k} - \alpha_{nk}. \tag{10}$$

This Lemma shows that the (horizontal) SIE subladders

$$\cdots \ \ker H_{n-1,k+1} \ \underset{-D}{\overset{k+1+XC}{\rightleftarrows}} \ \ker H_{nk} \ \underset{-D}{\overset{k+XC}{\rightleftarrows}} \ \ker H_{n+1,k-1} \ \cdots \tag{11}$$

with $\ker H_{nk} = \mathrm{eig}(A^{\sqsubset}_{nk}, \alpha_{nk})$ are well defined. Because of

$$\begin{aligned} D^{n+2}(k+XC) &= kD^{n+2} + (XD^{n+2} + (n+2)D^{n+1})C \\ &= [XCD + kD + (n+2)C]D^{n+1}D^n(-D) \ = \ -D^{n+1} \end{aligned}$$

we see that the even more restricted horizontal SIE subladders

$$\cdots \quad U_{n-1,k+1} \quad \underset{-D}{\overset{k+1+XC}{\rightleftarrows}} \quad U_{nk} \quad \underset{-D}{\overset{k+XC}{\rightleftarrows}} \quad U_{n+1,k-1} \quad \cdots \tag{12}$$

are well defined.

(2) Because of

$$\begin{aligned} & H_{n,k+1}(-C) - (-C)H_{nk} \\ = \ & [CXD + (k+1)D + n](-C) - \ (-C)[XCD + (k+1)D + n] \ = \ 0 \end{aligned}$$

$$\begin{aligned} & H_{nk}\mathcal{D}_n - \mathcal{D}_n H_{n,k+1} \\ = \ & [XCD + (k+1)D + n]\mathcal{D}_n - \mathcal{D}_n[CXD + (k+1)D + n] \\ = \ & XD(D^{n+1} - 1) - (D^{n+1} - 1)XD \\ = \ & (XD^{n+1} - D^{n+1}X)D \\ = \ & -(n+1)D^nD \ = \ 0 \end{aligned}$$

the restricted vertical ladders

$$\cdots \quad \ker H_{n-1,k} \quad \underset{\mathcal{D}_n}{\overset{-C}{\rightleftarrows}} \ \ker H_{nk} \ \underset{\mathcal{D}_{n+1}}{\overset{-C}{\rightleftarrows}} \quad \ker H_{n+1,k} \quad \cdots \tag{13}$$

and then, also the subladders

$$\cdots \quad U_{n-1,k} \quad \underset{\mathcal{D}_n}{\overset{-C}{\rightleftarrows}} \quad U_{nk} \quad \underset{\mathcal{D}_{n+1}}{\overset{-C}{\rightleftarrows}} \quad U_{n+1,k} \quad \cdots \tag{14}$$

are well defined. The vertical circuit endomorphisms on U_{nk} are

$$\begin{aligned} \mathcal{D}_n(-C) &= -D^{n+1} + 1 \ = \ 1 \\ (-C)\mathcal{D}_n &= -D^{n+1} + 1 \ = \ 1 \end{aligned}$$

This shows that (14) is an SIE subladder.

(3) The Northeast arrow in (9) is well defined. Starting at U_{nk}, then going East and North yields the operator

$$\begin{aligned}(-C)(k+XC) &= k+XC-(kD+DXC)\\ &= k+XC+n-[kD+(XD+1)C+n]\\ &= n+k+1+XC-[XCD+(k+1)D+n]\\ &= n+k+1+XC\end{aligned}$$

The Northeast narrow circuit endomorphism on U_{nk} is

$$\begin{aligned}&(n+1-XD)(n+1+k+XC)\\ =\ &(n+1)(n+1+k)+(n+1)XC-(n+1+k)XD-XDXC\\ =\ &(n+1)(n+1+k)-X[(n+1)+kD+(XD+1)C]\\ =\ &(n+1)(n+1+k)-XH_{nk} \ = \ (n+1)(n+1+k)\end{aligned}$$

(4) The Northeast-East triangle circuit endomorphism on U_{nk} is

$$\begin{aligned}&(-D)\mathcal{D}_{n+1}(n+k+1+XC)\\ =\ &-\mathcal{D}_{n+1}[(n+k+1)D+(XD+1)C]\\ =\ &-\mathcal{D}_{n+1}[XDC+(k+1)D+n+nC+C]\\ =\ &-\mathcal{D}_{n+1}R_{nk}-(n+1)\mathcal{D}_{n+1}C\\ =\ &-(n+1)(D^{n+2}-1)\\ =\ &n+1\end{aligned}$$

(5) The Northeast Square circuit endomorphism on U_{nk} is

$$\begin{aligned}&\mathcal{D}_n\,(-D)\,(-C)\,(k+XC)\\ =\ &\mathcal{D}_n\,C[kD+(XD+1)C]\\ =\ &(D^{n+1}-1)[XCD+(k+1)D+n-(n+1)]\\ =\ &-(n+1)(D^{n+1}-1)\\ =\ &n+1\end{aligned}$$

All the other statements can be shown in a similar fashion, alternatively one can refer to Lemma 2. ♦

8 Associated Laguerre polynomials

We have

$$U_{0,0} \;=\; \ker R_{0,0} \,\cap\, \ker D \;=\; \ker(XCD + D) \,\cap\, \ker D \;=\; \ker D.$$

Within the standard representation $V = \mathcal{C}^\infty(\mathbb{R})$ of the Weyl algebra we have $\ker D = \langle 1 \rangle$. Thus, for $n, k \geq 0$ the spaces $U_{nk} = \langle L_n^k \rangle$ are spanned by the associated Laguerre polynomials

$$L_n^k(x) \;=\; \sum_{j=0}^{n} \binom{n+k}{n-j} \frac{(-1)^j}{j!} x^j \;=\; \frac{(-1)^n}{n!}\, x^n + \ldots + \binom{n+k}{n}.$$

The diagram below shows the (monic) associated Laguerre polynomials in the grid for $n = 0, \ldots 5$ and $m = n + k = 0, \ldots, 7$. Note that all operators are only displayed up to scalar multiples. For clarity we did not draw the diagonal arrows.

$$
\begin{tikzcd}[column sep=small]
k=7\nearrow & \langle 1\rangle \arrow[r, shift left, "7+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_0"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x-7\rangle \arrow[r, shift left, "6+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_1"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^2-14x+42\rangle \arrow[r, shift left, "5+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_2"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^3-21x^2+126x-210\rangle \arrow[r, shift left, "4+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_3"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^4-28x^3+252x^2-840x+840\rangle \arrow[r, shift left, "3+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_4"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^5-35x^4+420x^3-2100x^2+4200x+2520\rangle \arrow[d, shift left, "\mathcal{D}_5"] \arrow[d, leftarrow, shift right, "-C"'] \\
k=6\nearrow & \langle 1\rangle \arrow[r, shift left, "6+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_0"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x-6\rangle \arrow[r, shift left, "5+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_1"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^2-12x+30\rangle \arrow[r, shift left, "4+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_2"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^3-18x^2+90x-120\rangle \arrow[r, shift left, "3+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_3"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^4-24x^3+180x^2-480x+360\rangle \arrow[r, shift left, "2+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_4"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^5-30x^4+300x^3-1200x^2+1800x-720\rangle \arrow[d, shift left, "\mathcal{D}_5"] \arrow[d, leftarrow, shift right, "-C"'] \\
k=5\nearrow & \langle 1\rangle \arrow[r, shift left, "5+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_0"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x-5\rangle \arrow[r, shift left, "4+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_1"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^2-10x+20\rangle \arrow[r, shift left, "3+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_2"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^3-15x^2+60x-60\rangle \arrow[r, shift left, "2+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_3"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^4-20x^3+120x^2-240x+120\rangle \arrow[r, shift left, "1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_4"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^5-25x^4+200x^3-600x^2+600x-120\rangle \arrow[d, shift left, "\mathcal{D}_5"] \arrow[d, leftarrow, shift right, "-C"'] \\
k=4\nearrow & \langle 1\rangle \arrow[r, shift left, "4+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_0"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x-4\rangle \arrow[r, shift left, "3+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_1"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^2-8x+12\rangle \arrow[r, shift left, "2+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_2"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^3-12x^2+36x-24\rangle \arrow[r, shift left, "1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_3"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^4-16x^3+72x^2-96x+24\rangle \arrow[r, shift left, "0+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_4"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^5-20x^4+120x^3-240x^2+120x\rangle \arrow[d, shift left, "\mathcal{D}_5"] \arrow[d, leftarrow, shift right, "-C"'] \\
k=3\nearrow & \langle 1\rangle \arrow[r, shift left, "3+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_0"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x-3\rangle \arrow[r, shift left, "2+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_1"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^2-6x+6\rangle \arrow[r, shift left, "1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_2"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^3-9x^2+18x-6\rangle \arrow[r, shift left, "0+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_3"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^4-12x^3+36x^2-24x\rangle \arrow[r, shift left, "-1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_4"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^5-15x^4+60x^3-60x^2\rangle \arrow[d, shift left, "\mathcal{D}_5"] \arrow[d, leftarrow, shift right, "-C"'] \\
k=2\nearrow & \langle 1\rangle \arrow[r, shift left, "2+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_0"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x-2\rangle \arrow[r, shift left, "1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_1"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^2-4x+2\rangle \arrow[r, shift left, "0+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_2"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^3-6x^2+6x\rangle \arrow[r, shift left, "-1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_3"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^4-8x^3+12x^2\rangle \arrow[r, shift left, "-2+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_4"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^5-10x^4+20x^3\rangle \arrow[d, shift left, "\mathcal{D}_5"] \arrow[d, leftarrow, shift right, "-C"'] \\
k=1\nearrow & \langle 1\rangle \arrow[r, shift left, "1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_0"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x-1\rangle \arrow[r, shift left, "0+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_1"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^2-2x\rangle \arrow[r, shift left, "-1+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_2"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^3-3x^2\rangle \arrow[r, shift left, "-2+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_3"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^4-4x^3\rangle \arrow[r, shift left, "-3+XC"] \arrow[r, leftarrow, shift right, "-D"'] \arrow[d, shift left, "\mathcal{D}_4"] \arrow[d, leftarrow, shift right, "-C"'] & \langle x^5-5x^4\rangle \arrow[d, shift left, "\mathcal{D}_5"] \arrow[d, leftarrow, shift right, "-C"'] \\
k=0\nearrow & \langle 1\rangle \arrow[r, shift left, "0+XC"] \arrow[r, leftarrow, shift right, "-D"'] & \langle x\rangle \arrow[r, shift left, "-1+XC"] \arrow[r, leftarrow, shift right, "-D"'] & \langle x^2\rangle \arrow[r, shift left, "-2+XC"] \arrow[r, leftarrow, shift right, "-D"'] & \langle x^3\rangle \arrow[r, shift left, "-3+XC"] \arrow[r, leftarrow, shift right, "-D"'] & \langle x^4\rangle \arrow[r, shift left, "-4+XC"] \arrow[r, leftarrow, shift right, "-D"'] & \langle x^5\rangle \\
 & n=0 & n=1 & n=2 & n=3 & n=4 & n=5
\end{tikzcd}
\tag{15}
$$

When taking into account the norming factors for the Laguerre polynomials the local diagram (9) has to be modified

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
L^{k+2}_{n-1} \arrow[r, shift left, "\frac{k+2+XC}{n}"] \arrow[d, shift left, "\mathcal{D}_{n-1}"] \arrow[dr, shift left, "\frac{(k+1)\mathcal{D}_{n-1}-X}{n}"'] & L^{k+1}_{n} \arrow[l, shift left, "-D"] \arrow[r, shift left, "\frac{k+1+XC}{n+1}"] \arrow[d, shift left, "\mathcal{D}_{n}"] & L^{k}_{n+1} \arrow[l, shift left, "-D"] \arrow[d, shift left, "\mathcal{D}_{n+1}"] \arrow[dl, shift left, "\frac{n+1-XD}{n+k+1}"] \\
L^{k+1}_{n-1} \arrow[u, shift left, "-C"] \arrow[r, shift left, "\frac{k+1+XC}{n}"] \arrow[d, shift left, "\mathcal{D}_{n-1}"] & L^{k}_{n} \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[ul, shift left, "DC"] \arrow[ur, shift left, "\frac{n+k+1+XC}{n+1}"] \arrow[r, shift left, "\frac{k+XC}{n+1}"] \arrow[d, shift left, "\mathcal{D}_{n}"] \arrow[dl, shift left, "\frac{n-XD}{n+k}"] \arrow[dr, shift left, "\frac{(k-1)\mathcal{D}_{n}-X}{n+1}"'] & L^{k-1}_{n+1} \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[d, shift left, "\mathcal{D}_{n+1}"] \\
L^{k}_{n-1} \arrow[u, shift left, "-C"] \arrow[r, shift left, "\frac{k+XC}{n}"] \arrow[ur, shift left, "\frac{n+k+XC}{n}"] & L^{k-1}_{n} \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[r, shift left, "\frac{k-1+XC}{n+1}"] & L^{k-2}_{n+1} \arrow[l, shift left, "-D"] \arrow[u, shift left, "-C"] \arrow[ul, shift left, "DC"]
\end{tikzcd}
\tag{16}
$$

The reader might wonder why we did not consider this diagram right from the start. But when including the norm factors the condition (10) on the scalar ladder commutator is not longer fulfilled. This condition is foundational for the whole “Laguerre system”.

We can now demonstrate that many of the classical identities for associated Laguerre polynomials can be derived from the diagram (16).

Theorem 5 (Associated Laguerre polynomials) *For the associated Laguerre polynomials the following relations hold*

(i) Three term recurrence relation.

$$(2n+1+k-X)L^k_n \;=\; (n+1)L^k_{n+1} + (n+k)L^k_{n-1}.$$

(ii) There are various so called three-point-rules

$$
\begin{aligned}
L^k_n &= \tfrac{1}{n}(n+k+1+XC)L^{k+1}_{n-1} - L^{k+1}_{n-1} &&= L^{k+1}_n - L^{k+1}_{n-1}\\
nL^k_n &= (n+k+XC)L^k_{n-1} &&= (n+k)L^k_{n-1} - XL^{k+1}_{n-1}\\
kL^k_n &= (k+XC)L^k_n - XCL^k_n &&= (n+1)L^{k-1}_{n+1} + XL^{k+1}_n\\
(n-X)L^k_n &= XCL^k_n + (n-XD)L^k_n &&= -XL^{k+1}_n + (n+k)L^k_{n-1}
\end{aligned}
$$

(iii) Sheffer sequence. The North and West arrows pointing to L_{n-1}^{k+1} yield the identity

$$DL_n^k \;=\; -L_{n-1}^{k+1} \;=\; (D-1)L_{n-1}^k.$$

(iv) Reflection across the main diagonal $k=0$.

$$L_{n+k}^{-k} \;=\; \frac{n!}{(n+k)!}\,(-X)^k L_n^k$$

(v) Rodrigues formula.

$$L_n^k(x) \;=\; \frac{1}{n!}X^{-k}C^n x^{n+k}$$

Within the standard representation we have

$$L_n^k(x) \;=\; \frac{1}{n!}x^{-k}e^x\partial^n(e^{-x}x^{n+k})$$

Proof (i) See the diagram (16). The two diagonal arrows in Northeast and Southwest direction starting at L_n^k contain the two equations

$$\begin{aligned}(n+1)L_{n+1}^k &= (n+1+k+XD-X)L_n^k\\ (n+k)L_{n-1}^k &= (n-XD)L_n^k.\end{aligned}$$

Adding up the two equations yields the three term recurrence relation in (i).

(ii) and (iii) were already shown in the theorem.

(iv) For fixed $n \in \mathbb{N}_0$ we prove by induction over $k \in \mathbb{N}_0$

$$U_{nk} \xrightarrow{(-X)^k} U_{n+k,-k}.$$

This operator jumps over $|k|$ steps in Southeast ($k \geq 0$) resp. Northwest ($k \leq 0$) direction.

The initial case $k=0$ is trivial. In order to show the induction step $k-1 \mapsto k$ we consider the following chain of operators

$$U_{nk} \xrightarrow{\mathcal{D}_n} U_{n,k-1} \xrightarrow{(-X)^{k-1}} U_{n+k-1,-(k-1)} \xrightarrow{-(k-1)+XC} U_{n+k,-k}$$

The operator on the middle arrow is well defined according to the induction hypothesis. Then on U_{nk} we compute the composition

$$\begin{aligned}
[-(k-1)+XC]\,(-X)^{k-1}\,\mathcal{D}_n &= X(-1)^{k-1}[-(k-1)X^{k-2}+CX^{k-1}]\mathcal{D}_n \\
&= X(-1)^{k-1}X^{k-1}\,C\,\mathcal{D}_n \\
&= -(-X)^k(D^{n+1}-1) \;=\; (-X)^k
\end{aligned}$$

For $k \leq 0$ the reflection is realized by the inverse multiplication operator $(-X)^k$.
Inspection of the leading coefficient then shows that

$$L_{n+k}^{-k} \quad = \quad \frac{n!}{(n+k)!}\,(-X)^k L_n^k.$$

(v) In order to compute $L_n^k \in U_{nk}$ we start at $U_{n+k,-(n+k)}$ (bottom line in the above diagram (15)), go k steps north, then reflect across the main diagonal, altogether

$$U_{n+k,-(n+k)} \xrightarrow{C^n} U_{n+k,-k} \xrightarrow{X^{-k}} U_{nk}$$

Since $U_{n+k,-(n+k)} = \langle x^{n+k}\rangle$ we find that

$$L_n^k(x) \quad = \quad \frac{1}{n!}X^{-k}C^n x^{n+k}$$

One can figure out the scalar factor by comparison of the leading coefficients.

Within the standard representation the operator $C = D - 1$ acts by $f(x) \mapsto e^x\partial[e^{-x}f(x)]$, hence the operator C^n acts by $f(x) \mapsto e^x\partial^n[e^{-x}f(x)]$. ♦

9 The Legendre-Gegenbauer grid

In the Weyl algebra we additionally define the operator

$$R \quad := \quad X^2 - 1$$

It fulfills the commutator relation

$$[D, R^j] \quad = \quad 2jXR^{j-1}, \qquad j \in \mathbb{N}_0. \tag{17}$$

For $n, \ell \in \mathbb{R}$ define the (second order differential) *Legendre operator*

$$H_{n\ell} \quad := \quad H_{n\ell}^{\mathrm{Leg}} \;:=\; RD^2 + (\ell+1)XD - n(n+\ell).$$

and its kernel

$$U_{n\ell} \quad := \quad \ker H_{n\ell} \;=\; \ker X^2 H_{n\ell} \;=\; \ker R H_{n\ell}.$$

Here we assume that the operators X^2 and R act as injective operators on the given representation space. This is fulfilled for the standard representation of the Weyl algebra on $\mathcal{C}^\infty(\mathbb{R})$ or any suitable subspace.

The diagram below shows the local structure of the *Legendre-Gegenbauer grid* in the Weyl algebra.

$$\begin{tikzcd}
U_{n-1,\ell+2} \arrow[r, shift left, "(n+\ell+1)X+RD"] \arrow[d, shift left, "R(XD-n+1)+\ell+1"] \arrow[dr, shift left, "(\ell+1)X+RD"'] & U_{n,\ell+2} \arrow[l, shift left, "nX-RD"] \arrow[r, shift left, "(n+\ell+2)X+RD"] \arrow[d, shift left, "R(XD-n)+\ell+1"] & U_{n+1,\ell+2} \arrow[l, shift left, "(n+1)X-RD"] \arrow[d, shift left, "R(XD-n-1)+\ell+1"] \\
U_{n-1,\ell} \arrow[u, shift left, "XD+n+\ell-1"] \arrow[r, shift left, "(n+\ell-1)X+RD"] \arrow[d, shift left, "R(XD-n+1)+\ell-1"] & U_{n,\ell} \arrow[ul, shift left, "D"] \arrow[u, shift left, "XD+n+\ell"] \arrow[l, shift left, "nX-RD"] \arrow[r, shift left, "(n+\ell)X+RD"] \arrow[d, shift left, "R(XD-n)+\ell-1"] \arrow[dr, shift left, "(\ell-1)X+RD"'] & U_{n+1,\ell} \arrow[u, shift left, "XD+n+\ell+1"] \arrow[l, shift left, "(n+1)X-RD"] \arrow[d, shift left, "R(XD-n-1)+\ell-1"] \\
U_{n-1,\ell-2} \arrow[u, shift left, "XD+n+\ell-3"] \arrow[r, shift left, "(n+\ell-3)X+RD"] & U_{n,\ell-2} \arrow[u, shift left, "XD+n+\ell-2"] \arrow[l, shift left, "nX-RD"] \arrow[r, shift left, "(n+\ell-2)X+RD"] & U_{n+1,\ell-2} \arrow[ul, shift left, "D"] \arrow[u, shift left, "XD+n+\ell-1"] \arrow[l, shift left, "(n+1)X-RD"]
\end{tikzcd} \tag{18}$$

Theorem 6 (Legendre-Gegenbauer grid) *The Legendre-Gegenbauer grid is a well defined SIE grid. The various circuit endomorphisms $U_{n\ell} \to U_{n\ell}$ act as scalars as follows*

$$
\begin{array}{llrcl}
\rightleftharpoons & \textit{East} & [(n+1)X - RD]\cdot[(n+\ell)X + RD] & = & (n+1)(n+\ell) \\
\rightleftharpoons & \textit{West} & [(n+\ell-1)X + RD]\cdot[nX - RD] & = & n(n+\ell-1) \\
\rightleftharpoons & \textit{North} & [R(XD-n)+(\ell+1)]\,(XD+n+\ell) & = & (n+\ell)(n+\ell+1) \\
\rightleftharpoons & \textit{South} & (XD+n+\ell-2)[R(XD-n)+(\ell-1)] & = & (n+\ell-2)(n+\ell-1) \\
\rightleftharpoons & \textit{Northwest} & [(\ell+1)X+RD]\,D & = & n(n+\ell) \\
\rightleftharpoons & \textit{Southeast} & D\,[(\ell-1)X+RD] & = & (n+1)(n+\ell-1) \\
\triangle & \textit{N-NW} & [(\ell+1)X+RD]\,(nX-RD)\,(XD+n+\ell) & = & n(n+\ell)(n+\ell+1) \\
\triangle & \textit{NW-N} & [R(XD-n)+\ell+1]\,[(n+\ell+1)X+RD]\,D & = & n(n+\ell)(n+\ell+1) \\
\triangle & \textit{NW-W} & [(n+\ell-1)X+RD]\,(XD+n+\ell+1)\,D & = & n(n+\ell-1)(n+\ell) \\
\triangle & \textit{W-NW} & [(\ell+1)X+RD]\,(XD+n+\ell-1)\,(nX-RD) & = & n(n+\ell-1)(n+\ell) \\
\square\circlearrowleft & & [R(XD-n)+\ell+1]\,[(n+1)X-RD] & & \\
 & & [XD+n+\ell+1]\,[(n+\ell)X+RD] & = & (n+1)(n+\ell) \\
 & & & & \quad (n+\ell+1)(n+\ell+2) \\
\square\circlearrowright & & [(n+1)X-RD]\,[R(XD-(n+1))+\ell+1] & & \\
 & & [(n+\ell+2)X+RD]\,(XD+n+\ell) & = & (n+1)(n+\ell) \\
 & & & & \quad (n+\ell+1)(n+\ell+2)
\end{array}
$$

Proof (1) The horizontal narrow circuit endomorphisms in East and West direction are:

$$
\begin{aligned}
A^{\sqsupset}_{n\ell} &= [(n+1)X - RD]\,[(n+\ell)X + RD] \\
&= (n+1)(n+\ell)X^2 - n(RDX - XRD) - RDRD \\
&\quad - \ell RDX + XRD \\
&= (n+1)(n+\ell)(R+1) - nR - R(RD+2X)D \\
&\quad - \ell R(XD+1) + XRD \\
&= -RH_{n\ell} + (n+1)(n+\ell)
\end{aligned}
$$

$$
\begin{aligned}
A^{\sqsubset}_{n\ell} &\quad [(n+\ell-1)X + RD]\,(nX - RD) \\
&= n(n+\ell-1)X^2 + nR[D,X] - RDRD - (\ell-1)XRD \\
&= n(n+\ell-1)(R+1) + nR - R(RD+2X)D - (\ell-1)XRD \\
&= -RH_{n\ell} + n(n+\ell-1)
\end{aligned}
$$

When choosing the number sequence $\alpha_{n\ell} = n(n+\ell-1)$, we find that the

commutator condition (7) in Lemma 3 is fulfilled,

$$A^{\sqsupset}_{n\ell} - A^{\sqsubset}_{n\ell} \quad = \quad (n+1)(n+\ell) - n(n+\ell-1) \; = \; \alpha_{n+1,\ell} - \alpha_{n\ell}. \tag{19}$$

Then the (horizontal) SIE subladders

$$\cdots \qquad U_{n-1,\ell} \quad \underset{nX\,-\,RD}{\overset{(n+\ell-1)X\,+\,RD}{\rightleftarrows}} \quad U_{n\ell} \quad \underset{(n+1)X\,-\,RD}{\overset{(n+\ell)X\,+\,RD}{\rightleftarrows}} \quad U_{n+1,\ell} \qquad \cdots \tag{20}$$

are well defined.

(2) The vertical narrow circuit endomorphisms in North and South direction are:

$$\begin{aligned}
B^{\sqsupset}_{n\ell} &= [R(XD-n)+(\ell+1)](XD+n+\ell) \\
&= RXDXD + (n+\ell)RXD - nRXD - n(n+\ell)R \\
&\quad + (\ell+1)XD + (\ell+1)(n+\ell) \\
&= RXDXD + \ell(X^2-1)XD - n(n+\ell)(X^2-1) \\
&\quad + (\ell+1)XD + (\ell+1)(n+\ell) \\
&= RX(XD+1)D + \ell X^3 D - n(n+\ell)X^2 \\
&\quad + XD + (n+\ell)(n+\ell+1) \\
&= X^2[RD^2 + (\ell+1)XD - n(n+\ell) - XD] \\
&\quad + RXD + XD + (n+\ell)(n+\ell+1) \\
&= X^2 H_{n\ell} + (n+\ell)(n+\ell+1)
\end{aligned}$$

$$\begin{aligned}
B^{\sqsubset}_{n\ell} &= (XD+n+\ell-2)[R(XD-n)+(\ell-1)] \\
&= X[D(RX)]D - nXDR + (\ell-1)XD + (n+\ell-2)RXD \\
&\qquad\quad -n(n+\ell-2)R + (n+\ell-2)(\ell-1) \\
&= X[(RX)D + (3X^2-1)]D - nX(RD+2X) \\
&\quad + (\ell-1)XD + (n+\ell-2)RXD \\
&\qquad\quad -n(n+\ell-2)(X^2-1) + (n+\ell-2)(\ell-1) \\
&= X^2RD^2 + 3X^3D + (\ell-2)XD + (\ell-2)RXD - 2nX^2 \\
&\qquad\quad -n(n+\ell-2)X^2 + (n+\ell-2)(\ell-1) \\
&= X^2 H_{n\ell} + (n+\ell-1)(n+\ell-2)
\end{aligned}$$

Again the difference of the two circuit operators is a scalar. With $\beta_{n\ell} = (n+\ell-1)(n+\ell-2)$ we find that the commutator condition (7) in Lemma 3 is fulfilled,

$$\begin{aligned}
B^{\sqsupset}_{n\ell} - B^{\sqsubset}_{n\ell} &= (n+\ell)(n+\ell+1) - (n+\ell-1)(n+\ell-2) \\
&= \beta_{n,\ell+2} - \beta_{n\ell}.
\end{aligned} \tag{21}$$

Lemma 3 implies that the vertical ladders in the Legendre-Gegenbauer grid

$$\cdots \qquad U_{n,\ell-2} \quad \underset{R(XD-n)+\ell-1}{\overset{XD+n+\ell-2}{\rightleftarrows}} U_{n\ell} \underset{R(XD-n)+\ell+1}{\overset{XD+n+\ell}{\rightleftarrows}} \quad U_{n,\ell+2} \qquad \cdots \tag{22}$$

are well defined.

(3) The diagonal narrow circuit endomorphisms in Southeast and Northwest direction are

$$\begin{aligned} D[(\ell-1)X+RD] &= (RD+2X)D+(\ell-1)(XD+1) \\ &= RD^2+(\ell+1)XD+\ell-1 \\ &= H_{n\ell}+(n+1)(n+\ell-1) \\ [(\ell+1)X+RD]D &= H_{n\ell}+n(n+\ell) \end{aligned}$$

Once more the difference of the two circuit operators is a scalar. With $\eta_{n\ell}=n(n+\ell)$ we find that the commutator condition (7) in Lemma 3 is fulfilled,

$$\begin{aligned} & D[(\ell-1)X+RD]-[(\ell+1)X+RD]D \\ = \;& (n+1)(n+\ell-1)-n(n+\ell) \\ = \;& \ell-1 \;=\; \eta_{n+1,\ell-2}-\eta_{n\ell}. \end{aligned} \tag{23}$$

Lemma 3 implies that the diagonal ladders in the Legendre-Gegenbauer grid

$$\cdots \qquad U_{n-1,\ell+2} \quad \underset{D}{\overset{(\ell+1)X+RD}{\rightleftarrows}} \quad U_{n\ell} \quad \underset{D}{\overset{(\ell-1)X+RD}{\rightleftarrows}} \quad U_{n+1,\ell-2} \qquad \cdots \tag{24}$$

are well defined SIE subladders.

(4) Note that the Northeast square is commutative.

$$\begin{aligned} & (XD+n+\ell+1)[(n+\ell)X+RD] \\ & \quad -[(n+\ell+2)X+RD](XD+n+\ell) \\ = \;& (n+\ell)XDX-(n+\ell+2)XXD+XDRD \\ & \quad -RDXD+(n+\ell+1)RD \\ & \quad -(n+\ell)RD+[(n+\ell+1)(n+\ell)-(n+\ell+2)(n+\ell)]X \\ = \;& (n+\ell)X(DX-XD)-2XD+X(RD+2X)D \\ & \qquad -R(XD+1)D+RD-(n+\ell)X \;=\; 0 \end{aligned}$$

Then, with steps (1) and (2) we get for the Northeast square endomorphism in counterclockwise direction

$$[R(XD-n)+\ell+1]\,[(n+1)X-RD]$$

$$
\begin{aligned}
& [XD+n+\ell+1]\,[(n+\ell)X+RD] \\
= \;& [R(XD-n)+\ell+1]\,[(n+1)X-RD] \\
& [(n+\ell+2)X+RD](XD+n+\ell) \\
= \;& [R(XD-n)+\ell+1]\,(n+1)(n+\ell+2)\,(XD+n+\ell) \\
= \;& (n+1)(n+\ell)(n+\ell+1)(n+\ell+2)
\end{aligned}
$$

and in clockwise direction

$$
\begin{aligned}
& [(n+1)X-RD]\,[R(XD-(n+1))+\ell+1] \\
& [(n+\ell+2)X+RD]\,(XD+n+\ell) \\
= \;& [(n+1)X-RD]\,[R(XD-(n+1))+\ell+1] \\
& (XD+n+\ell+1)\,[(n+\ell)X+RD] \\
= \;& [(n+1)X-RD]\,(n+\ell+1)(n+\ell+2)\,[(n+\ell)X+RD] \\
= \;& (n+1)(n+\ell)(n+\ell+1)(n+\ell+2)
\end{aligned}
$$

(5) The Northwest square on $U_{n\ell}$ is commutative up to scalar factors.

$$
\begin{aligned}
& (n+\ell-1)(nX-RD)(XD+n+\ell) \\
& \quad -(n+\ell+1)(XD+n+\ell-1)(nX-RD) \\
= \;& (n+\ell-1)[nX^2D+n(n+\ell)X-RDXD-(n+\ell)RD] \\
& \quad -(n+\ell+1)[nXDX+n(n+\ell-1)X-XDRD-(n+\ell-1)RD] \\
= \;& n(n+\ell-1)X^2D-n(n+\ell+1)XDX+[n(n+\ell)(n+\ell-1) \\
& \quad -n(n+\ell+1)(n+\ell-1)]X-(n+\ell-1)R(XD+1)D \\
& \quad +(n+\ell+1)X(RD+2X)D \\
& \quad +[-(n+\ell-1)(n+\ell)+(n+\ell+1)(n+\ell-1)]RD \\
= \;& n(n+\ell+1)X(XD-DX)-2nX^2D-n(n+\ell-1)X \\
& \quad +(n+\ell)(2X^2D-RD)+R(XD+1)D \\
& \quad +X(RD+2X)D+(n+\ell-1)RD \\
= \;& -2n(n+\ell)X-2nX^2D+(n+\ell)2X^2D \\
& \quad -RD+RXD^2+RD+XRD^2+2X^2D \\
= \;& 2X[RD^2+(\ell+1)XD-n(n+\ell)] \;=\; 2XH_{n\ell} \;=\; 0
\end{aligned}
$$

(6) The North Northwest triangle circuit endomorphism acts on $U_{n\ell}$ as follows

$$
\begin{aligned}
& [(\ell+1)X+RD]\,(nX-RD)(XD+n+\ell) \\
= \;& [(\ell+1)X+RD]\,[nX^2D+n(n+\ell)X-R(XD+1)D-(n+\ell)RD]
\end{aligned}
$$

$$
\begin{aligned}
&= [(\ell+1)X + RD]\Big[(-X)[RD^2 + (\ell+1)XD - n(n+\ell)] \\
&\qquad + (n+\ell+1)X^2D - (n+\ell+1)RD\Big] \\
&= [(\ell+1)X + RD]\,[-XH_{n\ell} + (n+\ell+1)D] \\
&= (n+\ell+1)[RD^2 + (\ell+1)XD - n(n+\ell)] \\
&\qquad + n(n+\ell)(n+\ell+1) - [(\ell+1)X + RD]XH_{n\ell} \\
&= [n+\ell+1 - (\ell+1)X^2 - RDX]H_{n\ell} + n(n+\ell)(n+\ell+1) \\
&= n(n+\ell)(n+\ell+1)
\end{aligned}
$$

We skip all the other calculations, since they can be done in a similar manner or by reference to Lemma 2. ♦

10 The Legendre-Gegenbauer polynomials

We define

$$W_{0,0} \;=\; \ker H_{0,0} \,\cap\, \ker D \;=\; \ker(XCD + D) \,\cap\, \ker D \;=\; \ker D.$$

This space generates an SIE subladder (W_{nk}) of the subladder (U_{nk}). Within the standard representation $V = \mathcal{C}^\infty(\mathbb{R})$ of the Weyl algebra we have $\ker D = \langle 1 \rangle$. Thus, for $n, k \geq 0$ the spaces W_{nk} are one–dimensional. For $n \geq 0$ the one–dimensional spaces $W_{n\ell}$ exactly contain the Legendre-Gegenbauer polynomials

$$
\begin{aligned}
P_n^\ell(x) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{\Gamma(n + \frac{\ell}{2} - j)}{\Gamma(\frac{\ell}{2}) j!(n-2j)!} (2x)^{n-2j} \\
&= \frac{(\ell+2n-2)(\ell+2n-4)\cdot \ldots \cdot \ell}{n!}\, x^n \\
&\quad - \frac{(\ell+2n-4)(\ell+2n-6)\cdot \ldots \cdot \ell}{(n-2)!}\, x^{n-2} \pm \ldots
\end{aligned}
$$

The Legendre-Gegenbauer polynomials with small index are shown in the following diagram

$$
\begin{array}{ccccccccccccc}
\ell=13 & \langle 1\rangle & \underset{1X-RD}{\overset{13X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{14X+RD}{\rightleftarrows}} & \langle 15x^2-1\rangle & \underset{3X-RD}{\overset{15X+RD}{\rightleftarrows}} & \langle 17x^3-3x\rangle & \underset{4X-RD}{\overset{16X+RD}{\rightleftarrows}} & \langle 323x^4-102x^2+3\rangle & \underset{5X-RD}{\overset{17X+RD}{\rightleftarrows}} & \langle 399x^5-190x^3+15x\rangle \\
 & {\scriptstyle XD+11}\big\uparrow\big\downarrow{\scriptstyle R(XD)+12} & & {\scriptstyle XD+12}\big\uparrow\big\downarrow{\scriptstyle R(XD-1)+12} & & {\scriptstyle XD+13}\big\uparrow\big\downarrow{\scriptstyle R(XD-2)+12} & & {\scriptstyle XD+14}\big\uparrow\big\downarrow{\scriptstyle R(XD-3)+12} & & {\scriptstyle XD+15}\big\uparrow\big\downarrow{\scriptstyle R(XD-4)+12} & & {\scriptstyle XD+16}\big\uparrow\big\downarrow{\scriptstyle R(XD-5)+12} \\
\ell=11 & \langle 1\rangle & \underset{1X-RD}{\overset{11X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{12X+RD}{\rightleftarrows}} & \langle 13x^2-1\rangle & \underset{3X-RD}{\overset{13X+RD}{\rightleftarrows}} & \langle 15x^3-3x\rangle & \underset{4X-RD}{\overset{14X+RD}{\rightleftarrows}} & \langle 85x^4-30x^2+1\rangle & \underset{5X-RD}{\overset{15X+RD}{\rightleftarrows}} & \langle 323x^5-170x^3+15x\rangle \\
 & {\scriptstyle XD+9}\big\uparrow\big\downarrow{\scriptstyle R(XD)+10} & & {\scriptstyle XD+10}\big\uparrow\big\downarrow{\scriptstyle R(XD-1)+10} & & {\scriptstyle XD+11}\big\uparrow\big\downarrow{\scriptstyle R(XD-2)+10} & & {\scriptstyle XD+12}\big\uparrow\big\downarrow{\scriptstyle R(XD-3)+10} & & {\scriptstyle XD+13}\big\uparrow\big\downarrow{\scriptstyle R(XD-4)+10} & & {\scriptstyle XD+14}\big\uparrow\big\downarrow{\scriptstyle R(XD-5)+10} \\
\ell=9 & \langle 1\rangle & \underset{1X-RD}{\overset{9X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{10X+RD}{\rightleftarrows}} & \langle 11x^2-1\rangle & \underset{3X-RD}{\overset{11X+RD}{\rightleftarrows}} & \langle 13x^3-3x\rangle & \underset{4X-RD}{\overset{12X+RD}{\rightleftarrows}} & \langle 65x^4-26x^2+1\rangle & \underset{5X-RD}{\overset{13X+RD}{\rightleftarrows}} & \langle 17x^5-10x^3+x\rangle \\
 & {\scriptstyle XD+7}\big\uparrow\big\downarrow{\scriptstyle R(XD)+8} & & {\scriptstyle XD+8}\big\uparrow\big\downarrow{\scriptstyle R(XD-1)+8} & & {\scriptstyle XD+9}\big\uparrow\big\downarrow{\scriptstyle R(XD-2)+8} & & {\scriptstyle XD+10}\big\uparrow\big\downarrow{\scriptstyle R(XD-3)+8} & & {\scriptstyle XD+11}\big\uparrow\big\downarrow{\scriptstyle R(XD-4)+8} & & {\scriptstyle XD+12}\big\uparrow\big\downarrow{\scriptstyle R(XD-5)+8} \\
\ell=7 & \langle 1\rangle & \underset{1X-RD}{\overset{7X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{8X+RD}{\rightleftarrows}} & \langle 9x^2-1\rangle & \underset{3X-RD}{\overset{9X+RD}{\rightleftarrows}} & \langle 11x^3-3x\rangle & \underset{4X-RD}{\overset{10X+RD}{\rightleftarrows}} & \langle 143x^4-66x^2+3\rangle & \underset{5X-RD}{\overset{11X+RD}{\rightleftarrows}} & \langle 39x^5-26x^3+3x\rangle \\
 & {\scriptstyle XD+5}\big\uparrow\big\downarrow{\scriptstyle R(XD)+6} & & {\scriptstyle XD+6}\big\uparrow\big\downarrow{\scriptstyle R(XD-1)+6} & & {\scriptstyle XD+7}\big\uparrow\big\downarrow{\scriptstyle R(XD-2)+6} & & {\scriptstyle XD+8}\big\uparrow\big\downarrow{\scriptstyle R(XD-3)+6} & & {\scriptstyle XD+9}\big\uparrow\big\downarrow{\scriptstyle R(XD-4)+6} & & {\scriptstyle XD+10}\big\uparrow\big\downarrow{\scriptstyle R(XD-5)+6} \\
\ell=5 & \langle 1\rangle & \underset{1X-RD}{\overset{5X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{6X+RD}{\rightleftarrows}} & \langle 7x^2-1\rangle & \underset{3X-RD}{\overset{7X+RD}{\rightleftarrows}} & \langle 9x^3-3x\rangle & \underset{4X-RD}{\overset{8X+RD}{\rightleftarrows}} & \langle 33x^4-18x^2+1\rangle & \underset{5X-RD}{\overset{9X+RD}{\rightleftarrows}} & \langle 143x^5-110x^3+15x\rangle \\
 & {\scriptstyle XD+3}\big\uparrow\big\downarrow{\scriptstyle R(XD)+4} & & {\scriptstyle XD+4}\big\uparrow\big\downarrow{\scriptstyle R(XD-1)+4} & & {\scriptstyle XD+5}\big\uparrow\big\downarrow{\scriptstyle R(XD-2)+4} & & {\scriptstyle XD+6}\big\uparrow\big\downarrow{\scriptstyle R(XD-3)+4} & & {\scriptstyle XD+7}\big\uparrow\big\downarrow{\scriptstyle R(XD-4)+4} & & {\scriptstyle XD+8}\big\uparrow\big\downarrow{\scriptstyle R(XD-5)+4} \\
\ell=3 & \langle 1\rangle & \underset{1X-RD}{\overset{3X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{4X+RD}{\rightleftarrows}} & \langle 5x^2-1\rangle & \underset{3X-RD}{\overset{5X+RD}{\rightleftarrows}} & \langle 7x^3-3x\rangle & \underset{4X-RD}{\overset{6X+RD}{\rightleftarrows}} & \langle 21x^4-14x^2+1\rangle & \underset{5X-RD}{\overset{7X+RD}{\rightleftarrows}} & \langle 231x^5-210x^3+35x\rangle \\
 & {\scriptstyle XD+1}\big\uparrow\big\downarrow{\scriptstyle R(XD)+2} & & {\scriptstyle XD+2}\big\uparrow\big\downarrow{\scriptstyle R(XD-1)+2} & & {\scriptstyle XD+3}\big\uparrow\big\downarrow{\scriptstyle R(XD-2)+2} & & {\scriptstyle XD+4}\big\uparrow\big\downarrow{\scriptstyle R(XD-3)+2} & & {\scriptstyle XD+5}\big\uparrow\big\downarrow{\scriptstyle R(XD-4)+2} & & {\scriptstyle XD+6}\big\uparrow\big\downarrow{\scriptstyle R(XD-5)+2} \\
\ell=1 & \langle 1\rangle & \underset{1X-RD}{\overset{1X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{2X+RD}{\rightleftarrows}} & \langle 3x^2-1\rangle & \underset{3X-RD}{\overset{3X+RD}{\rightleftarrows}} & \langle 5x^3-3x\rangle & \underset{4X-RD}{\overset{4X+RD}{\rightleftarrows}} & \langle 35x^4-30x^2+3\rangle & \underset{5X-RD}{\overset{5X+RD}{\rightleftarrows}} & \langle 63x^5-70x^3+15x\rangle \\
 & {\scriptstyle XD-1}\big\uparrow\big\downarrow{\scriptstyle R(XD)} & & {\scriptstyle XD}\big\uparrow\big\downarrow{\scriptstyle R(XD-1)} & & {\scriptstyle XD+1}\big\uparrow\big\downarrow{\scriptstyle R(XD-2)} & & {\scriptstyle XD+2}\big\uparrow\big\downarrow{\scriptstyle R(XD-3)} & & {\scriptstyle XD+3}\big\uparrow\big\downarrow{\scriptstyle R(XD-4)} & & {\scriptstyle XD+4}\big\uparrow\big\downarrow{\scriptstyle R(XD-5)} \\
\ell=-1 & \langle 1\rangle & \underset{1X-RD}{\overset{-1X+RD}{\rightleftarrows}} & \langle x\rangle & \underset{2X-RD}{\overset{0X+RD}{\rightleftarrows}} & \langle x^2-1\rangle & \underset{3X-RD}{\overset{1X+RD}{\rightleftarrows}} & \langle 3x^3-3x\rangle & \underset{4X-RD}{\overset{2X+RD}{\rightleftarrows}} & \langle 5x^4-6x^2+1\rangle & \underset{5X-RD}{\overset{3X+RD}{\rightleftarrows}} & \langle 7x^5-10x^3+3x\rangle \\
 & n=0 & & n=1 & & n=2 & & n=3 & & n=4 & & n=5
\end{array}
\tag{25}
$$

Now taking into account the norming factors for the Gegenbauer polynomials the local diagram (18) has to be modified

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
P_{n-1}^{\ell+2} \arrow[r, shift left, "\frac{(n+\ell+1)X+RD}{n}"] \arrow[d, shift left, "\frac{\ell[R(XD-n+1)+\ell+1]}{(n+\ell-1)(n+\ell)}"] \arrow[dr, "\frac{\ell[(\ell+1)X+RD]}{n(n+\ell)}"'] & P_{n}^{\ell+2} \arrow[l, shift left, "\frac{nX-RD}{n+\ell+1}"] \arrow[r, shift left, "\frac{(n+\ell+2)X+RD}{n+1}"] \arrow[d, shift left, "\frac{\ell[R(XD-n)+\ell+1]}{(n+\ell)(n+\ell+1)}"] & P_{n+1}^{\ell+2} \arrow[l, shift left, "\frac{(n+1)X-RD}{n+\ell+2}"] \arrow[d, shift left, "\frac{\ell[R(XD-n-1)+\ell+1]}{(n+\ell+1)(n+\ell+2)}"] \\
P_{n-1}^{\ell} \arrow[u, shift left, "\frac{XD+n+\ell-1}{\ell}"] \arrow[r, shift left, "\frac{(n+\ell-1)X+RD}{n}"] \arrow[d, shift left, "\frac{(\ell-2)[R(XD-n+1)+\ell-1]}{(n+\ell-3)(n+\ell-2)}"] & P_{n}^{\ell} \arrow[ul, "\frac{D}{\ell}"] \arrow[u, shift left, "\frac{XD+n+\ell}{\ell}"] \arrow[l, shift left, "\frac{nX-RD}{n+\ell-1}"] \arrow[r, shift left, "\frac{(n+\ell)X+RD}{n+1}"] \arrow[d, shift left, "\frac{(\ell-2)[R(XD-n)+\ell-1]}{(n+\ell-2)(n+\ell-1)}"] \arrow[dr, "\frac{(\ell-2)[(\ell-1)X+RD]}{(n+1)(n+\ell-1)}"'] & P_{n+1}^{\ell} \arrow[u, shift left, "\frac{XD+n+\ell+1}{\ell}"] \arrow[l, shift left, "\frac{(n+1)X-RD}{n+\ell}"] \arrow[d, shift left, "\frac{(\ell-2)[R(XD-n-1)+\ell-1]}{(n+\ell-1)(n+\ell)}"] \\
P_{n-1}^{\ell-2} \arrow[u, shift left, "\frac{XD+n+\ell-3}{\ell-2}"] \arrow[r, shift left, "\frac{(n+\ell-3)X+RD}{n}"] & P_{n}^{\ell-2} \arrow[u, shift left, "\frac{XD+n+\ell-2}{\ell-2}"] \arrow[l, shift left, "\frac{nX-RD}{n+\ell-3}"] \arrow[r, shift left, "\frac{(n+\ell-2)X+RD}{n+1}"] & P_{n+1}^{\ell-2} \arrow[u, shift left, "\frac{XD+n+\ell-1}{\ell-2}"] \arrow[l, shift left, "\frac{(n+1)X-RD}{n+\ell-2}"] \arrow[ul, "\frac{D}{\ell-2}"]
\end{tikzcd}
\tag{26}
$$

The norm factors were not incorporated in the first Legendre-Gegenbauer diagram (18), because then the scalar commutator conditions (19), (21) and (23) are destroyed. We can read off some of the classical identities for Legendre-Gegenbauer polynomials from the diagram (26).

Theorem 7 (Legendre-Gegenbauer polynomials) *For the Legendre-Gegenbauer polynomials the following relations hold*

(i) Three term recurrence relation.

$$(n+1)P_{n+1}^{\ell} + (n+\ell-1)P_{n-1}^{\ell} \quad = \quad (2n+\ell)XP_n^{\ell}.$$

(ii) There are various so called three-point-rules

$$
\begin{aligned}
(n+\ell)P_n^{\ell} &= (XD+n+\ell)P_n^{\ell} - XDP_n^{\ell} \\
&= \ell P_n^{\ell+2} - \ell XP_{n-1}^{\ell+2}
\end{aligned}
$$

$$nXP_n^{\ell} \quad = \quad (nX-RD)P_n^{\ell} + RDP_n^{\ell}$$

$$
\begin{aligned}
&= (n+\ell-1)P^{\ell}_{n-1} + \ell R P^{\ell+2}_{n-1} \\
(n+\ell)P^{\ell}_{n} &= (n+\ell+RD)P^{\ell}_{n} - RDP^{\ell}_{n} \\
&= (n+1)P^{\ell}_{n+1} - \ell R P^{\ell+2}_{n-1} \\
(\ell-1)(\ell-2)XP^{\ell}_{n} &= (\ell-2)[(\ell-1)X+RD]P^{\ell}_{n} - (\ell-2)RDP^{\ell}_{n} \\
&= (n+1)(n+\ell-1)P^{\ell-2}_{n+1} - \ell(\ell-2)RP^{\ell+2}_{n-1}
\end{aligned}
$$

(iii) Rodrigues formula. For an index pair $(n,\ell) \in \mathbb{N} \times (2\mathbb{N}+1)$ the Legendre-Gegenbauer polynomial P^{ℓ}_{n} is given by

$$
P^{\ell}_{n} \quad = \quad \frac{(\ell+2n-2)\cdot(\ell+2n-4)\cdot\ldots\cdot\ell}{(\ell+2n-1)\cdot\ldots\cdot(\ell+n)\cdot n!}\, R^{-\frac{\ell-1}{2}}\, D^{n} R^{n+\frac{\ell-1}{2}}\, 1
$$

Proof (i) By looking at the two horizontal arrows in East and West direction starting at P^{ℓ}_{n} in (26) we get the two equations

$$
\begin{aligned}
(n+1)P^{\ell}_{n+1} &= [(n+\ell)X + RD)P^{\ell}_{n} \\
(n+\ell-1)P^{\ell}_{n-1} &= [nX - RD)P^{\ell}_{n}.
\end{aligned}
$$

Adding up the two equations yields the three term recurrence relation.

(iii) (1) We are going to show the two following equalities

$$
\begin{aligned}
P^{\ell}_{n} &= \frac{\ell\cdot(RD+(\ell+1)X)}{n(n+\ell)} \cdot \frac{(\ell+2)\cdot(RD+(\ell+3)X)}{(n-1)(n+\ell+1)} \cdot \ldots \\
&\qquad \ldots \cdot \frac{(\ell+2n-2)\cdot(RD+(\ell+2n-1)X)}{1\cdot(2n+\ell-1)}\, 1 \qquad (27) \\
&= \frac{(\ell+2n-2)\cdot(\ell+2n-4)\cdot\ldots\cdot\ell}{(\ell+2n-1)!}\, D^{n+\ell-1}\, R^{n+\frac{\ell-1}{2}}\, 1 \qquad (28) \\
&= \frac{(\ell+2n-2)\cdot(\ell+2n-4)\cdot\ldots\cdot\ell}{(\ell+2n-1)\cdot\ldots\cdot(\ell+n)\cdot n!}\, R^{-\frac{\ell-1}{2}}\, D^{n} R^{n+\frac{\ell-1}{2}}\, 1
\end{aligned}
$$

(2) The first equation is clear, since the operator on the right hand side belongs to the n step southeast diagonal path

$$
1 \;=\; P^{\ell+2n}_{0} \longrightarrow P^{\ell+2n-2}_{1} \longrightarrow \ldots \longrightarrow P^{\ell}_{n}
$$

See the diagrams (25) and (26).

(3) We prove the second equation for $\ell = 1$. It is valid within the Weyl algebra, independent of the representation. Since this is clear for the scalar factors it remains to prove by induction over $n \in \mathbb{N}$ that

$$(RD + 2X) \cdot (RD + 4X) \cdot \ldots \cdot (RD + 2nX) \; = \; D^n R^n$$

For $n = 1$ we have

$$RD + 2X \quad = \quad DR.$$

and then

$$\begin{aligned} & (RD + 2X) \cdot (RD + 4X) \cdot \ldots \cdot (RD + 2nX) \cdot (RD + 2(n+1)X) \\ = \;& D^n R^n \cdot (RD + 2(n+1)X) \\ = \;& D^n (R^{n+1} D + 2(n+1) X R^n) \\ = \;& D^{n+1} R^{n+1} \end{aligned}$$

(4) We now perform the induction step $(n, \ell) \to (n-1, \ell+2)$. Here we have

$$\begin{aligned} P_{n-1}^{\ell+2} \; = \; \frac{D}{\ell} P_n^\ell \quad &= \quad \frac{D}{\ell} \frac{(\ell+2n-2)(\ell+2n-4) \cdot \ldots \cdot \ell}{(\ell+2n-1)!} D^{n+\ell-1} R^{\frac{\ell+2n-1}{2}} \, 1 \\ &= \quad \frac{(\ell+2n-2)(\ell+2n-4) \cdot \ldots \cdot (\ell+2)}{(\ell+2n-1)!} D^{n+\ell} R^{\frac{\ell+2n-1}{2}} \, 1. \end{aligned}$$

So far we have proved the two upper lines in (28) for all $n \in \mathbb{N}$, $\ell \in 2\mathbb{N}+1$.

(5) We prove the third identity in (28), it is

$$\frac{1}{(n+\ell-1)!} R^{\frac{\ell-1}{2}} D^{n+\ell-1} R^{n+\frac{\ell-1}{2}} \, 1 \quad = \quad \frac{1}{n!} D^n R^{n+\frac{\ell-1}{2}} \, 1.$$

This is an equation between two polynomials, we check it by applying the derivative operator r times for all $r = 0, \ldots, n$

$$\frac{1}{(n+\ell-1)!} D^r R^{\frac{\ell-1}{2}} D^{n+\ell-1} R^{n+\frac{\ell-1}{2}} \, 1 \quad = \quad \frac{1}{n!} D^{n+r} R^{n+\frac{\ell-1}{2}} \, 1.$$

and then comparing the leading coefficient. In fact it is

$$\frac{(2n+\ell-1)!}{n! \cdot (n+\ell-1-r)!}$$

on both sides. So our claim follows. ♦

11 The h-Weyl algebra

Now for fixed $h \neq 0$ we consider the so called *h-Weyl algebra*. It is generated as an associative unital $\mathbb{C}$ algebra by three operators M, D, X with the four relations

$$[M, D] = 0, \quad [M, X] = h^2 D, \quad [D, X] = M, \quad M^2 - h^2 D^2 = 1. \tag{29}$$

It is not hard to derive the following additional relation

$$DXM - MXD \;=\; 1 \tag{30}$$

The h–Weyl algebra has two particular representations. The first one, called diff representation is on a space of functions (or appropriate subspace) with a discrete variable

$$\mathcal{F}(h\mathbb{Z}, \mathbb{C}), \qquad \begin{cases} Mf(x) & := & \frac{f(x+h)+f(x-h)}{2} \\ Df(x) & := & \frac{f(x+h)-f(x-h)}{2h} \\ Xf(x) & := & x \cdot f(x). \end{cases} \tag{31}$$

The symbols M (mix), D (difference) and X arise from this difference operator representation. The second representation, called trig representation, is related to the first one by Fourier transform and Pontryagin duality. The h–Weyl algebra acts on smooth functions on a circle with radius $\frac{1}{h}$ ($= \frac{2\pi}{h}$–periodic functions on $\mathbb{R}$) (or appropriate subspace)

$$\mathcal{C}^\infty(\tfrac{1}{h}\mathbb{S}, \mathbb{C}), \qquad \begin{cases} Mf(x) & := & \cos(hx)f(x) \\ Df(x) & := & \frac{i\sin(hx)}{h}f(x) \\ Xf(x) & := & if'(x). \end{cases} \tag{32}$$

For $M = I$ and $h = 0$ the h–Weyl algebra reduces to the original Weyl algebra. The two representations reduce to the standard representation of the Weyl algebra.

12 The Binomial grid

From now on we will only consider the h-Weyl algebra with $h = 1$. We define the following operators

$$\begin{aligned} G_j &:= jD + MX = (j+1)D + XM \\ L_j &:= jM + DX = (j+1)M + XD \end{aligned}$$

and note some relations

$$\begin{aligned} L_j D - D L_{j-1} &= (jM + DX)D - D(jM + XD) \\ &= 0 \end{aligned} \tag{33}$$

$$\begin{aligned} G_j M - M G_{j-1} &= (jD + MX)M - M(jD + XM) \\ &= 0 \end{aligned} \tag{34}$$

$$\begin{aligned} G_{j-1} L_j - L_{j-1} G_j &= (jD + XM)(jM + DX) - (jM + XD)(jD + MX) \\ &= jX(M^2 - D^2) + j(D^2 - M^2)X \\ &= 0 \end{aligned} \tag{35}$$

$$\begin{aligned} L_{j-1} L_j - G_{j-1} G_j &= (jM + XD)(jM + DX) - (jD + XM)(jD + MX) \\ &= j^2(M^2 - D^2) + X(D^2 - M^2)X \\ &\qquad + j(MDX - DMX + XDM - XMD) \\ &= j^2 - X^2 \end{aligned} \tag{36}$$

Define for $n, m \in \mathbb{Z}$ the (second order difference) *binomials operators*

$$\begin{aligned} H_{nm} := H_{nm}^{\mathrm{bin}} &:= (n + m + 1)M^2 + DXM - (n + 1) \\ &\overset{(30)}{=} (n + m + 1)M^2 + MXD - n \\ &= (n + m + 1)(M^2 - 1) + MXD + (m + 1) \\ &= (n + m + 1)D^2 + MXD + (m + 1) \\ &\overset{(30)}{=} (n + m + 1)D^2 + DXM + m \end{aligned} \tag{37}$$

in the 1-Weyl algebra and then the spaces

$$U_{nm} := \ker H_{nm}$$

with respect to any representation. The following diagram shows the local structure of the binomial grid.

$$
\begin{tikzcd}
U_{n-1,m+1} \arrow[r, shift left, "M"] \arrow[d, shift left, "G_{n+m}"] & U_{n,m+1} \arrow[l, shift left, "L_{n+m+1}"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_{n+m+1}"] & U_{n+1,m+1} \arrow[l, shift left, "L_{n+m+2}"] \arrow[d, shift left, "G_{n+m+2}"] \\
U_{n-1,m} \arrow[u, shift left, "-D"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_{n+m-1}"] & U_{n,m} \arrow[u, shift left, "-D"] \arrow[l, shift left, "L_{n+m}"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_{n+m}"] & U_{n+1,m} \arrow[u, shift left, "-D"] \arrow[l, shift left, "L_{n+m+1}"] \arrow[d, shift left, "G_{n+m+1}"] \\
U_{n-1,m-1} \arrow[u, shift left, "-D"] \arrow[r, shift left, "M"] & U_{n,m-1} \arrow[u, shift left, "-D"] \arrow[l, shift left, "L_{n+m-1}"] \arrow[r, shift left, "M"] & U_{n+1,m-1} \arrow[u, shift left, "-D"] \arrow[l, shift left, "L_{n+m}"]
\end{tikzcd}
\tag{38}
$$

Theorem 8 (Binomial grid) *The Binomial grid is a well defined SIE grid. The various circuit endomorphisms $U_{nm} \to U_{nm}$ act as scalars as follows*

$$
\begin{array}{llrcl}
\rightleftharpoons & \textit{East} & L_{n+m+1}M & = & n+1 \\
\rightleftharpoons & \textit{West} & ML_{n+m} & = & n \\
\rightleftharpoons & \textit{North} & G_{n+m+1}(-D) & = & m+1 \\
\rightleftharpoons & \textit{South} & (-D)G_{n+m} & = & m \\
\square\,\circlearrowleft & & G_{n+m+1}\,L_{n+m+2}\,(-D)\,M & = & (n+1)(m+1) \\
\square\,\circlearrowright & & L_{n+m+1}\,G_{n+m+2}\,M\,(-D) & = & (n+1)(m+1)
\end{array}
$$

Proof (1) The horizontal narrow circuit endomorphisms are

$$
\begin{aligned}
A^{\sqsupset}_{nm} &= L_{n+m+1}M = (n+m+1)M^2 + DXM = H_{nm} + (n+1) \\
A^{\sqsubset}_{nm} &= ML_{n+m} = (n+m+1)M^2 + MXD = H_{nm} + n
\end{aligned}
$$

When choosing the number sequence $\alpha_{nm} = n$, we find that the commutator condition (7) in Lemma 3 is fulfilled,

$$
A^{\sqsupset}_{nm} - A^{\sqsubset}_{nm} = 1 = \alpha_{n+1,m} - \alpha_{nm}. \tag{39}
$$

Lemma 3 shows that the horizontal ladders

$$\cdots \quad U_{n-1,m} \quad \underset{L_{n+m}}{\overset{M}{\rightleftarrows}} \quad U_{nm} \quad \underset{L_{n+m+1}}{\overset{M}{\rightleftarrows}} \quad U_{n+1,m} \quad \cdots \tag{40}$$

in the binomial grid (38) are well defined SIE subladders. Within the diff representation (31) the operator M is the discrete heat distribution operator. So the above ladder can be called the Heat ladder.

(2) The vertical circuit endomorphisms in North and South direction are:

$$\begin{aligned} B^{\sqsupset}_{n\ell} &= G_{n+m+1}(-D) \;=\; -(n+m+1)D^2 - MXD \\ &= -H_{nm} + (m+1) \\ B^{\sqsubset}_{n\ell} &= (-D)G_{n+m} \;=\; -(n+m+1)D^2 - DXM \\ &= -H_{nm} + m \end{aligned}$$

The difference of the two circuit endomorphisms is again a scalar,

$$B^{\sqsupset}_{nm} - B^{\sqsubset}_{nm} \quad = \quad 1 \;=\; \beta_{n,m+1} - \beta_{nm}, \tag{41}$$

where $\beta_{nm} = m$. Lemma 3 again implies that the vertical ladders

$$\cdots \quad U_{n,m-1} \quad \underset{G_{n+m}}{\overset{-D}{\rightleftarrows}} \quad U_{nm} \quad \underset{G_{n+m+1}}{\overset{-D}{\rightleftarrows}} \quad U_{n,m+1} \quad \cdots \tag{42}$$

in the binomial grid (38) are well defined.

(3) We use (33), (34) and steps (1) and (2) in order to compute the North-east square circuit endomorphism on U_{nm} in counterclockwise and clockwise direction

$$\begin{aligned} G_{n+m+1}\, L_{n+m+2}\, (-D)\, M &= G_{n+m+1}\, (-D)\, L_{n+m+1}\, M \;=\; (n+1)(m+1) \\ L_{n+m+1}\, G_{n+m+2}\, M\, (-D) &= L_{n+m+1}\, M\, G_{n+m+1}\, (-D) \;=\; (n+1)(m+1) \end{aligned}$$

The proof is finished. ♦

13 The Binomials

The relations

$$\begin{aligned} G_0 &= MX \\ L_0 &= DX \\ X &= (M+D)(M-D)X \;=\; (M+D)(G_0 - L_0) \\ H_{00} &= ML_0 \;=\; MDX \end{aligned}$$

see (29), show that

$$W_{00} \;:=\; \ker X \;=\; \ker G_0 \cap \ker L_0 \;\subseteq \ker H_{00} \;=\; U_{00}. \tag{43}$$

This subspace of U_{00} generates an SIE subrepresentation (W_{nm}) of the grid (U_{nm}) by

$$W_{nm} \;:=\; M^n D^m(W_{00}).$$

Within the trigonometric representation (32) we have $W_{00} = \ker X = \langle 1 \rangle$, where 1 is the constant–1 function on $\mathbb{S}$. So the subrepresentation (W_{nm}) is given by

$$W_{nm} \;\;=\;\; \langle \cos^n(x)\; \sin^m(x) \rangle.$$

Within the Diff representation (31) we have $W_{00} = \ker X = \langle \delta_0 \rangle$, where δ_z is the "Kronecker Delta" function on $\mathbb{Z}$. The representation spaces (W_{nm}) contain centered binomial functions with alternating signs. The horizontal ladder (W_{n0}) is the ladder of the classical centered binomials. They are related to the discrete Harmonic Oscillator [5]. This representation (W_{nm})

is displayed in the following last diagram

$$
\begin{tikzcd}
m{=}4 & \begin{array}{l}\langle\delta_{-4}-3\delta_{-2}\\ +6\delta_0-4\delta_2+\delta_4\rangle\end{array} \arrow[r, shift left, "M"] \arrow[d, shift left, "G_4"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-5}-3\delta_{-3}\\ +2\delta_{-1}+2\delta_1\\ -3\delta_3+\delta_5\rangle\end{array} \arrow[l, shift left, "L_5"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_5"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-6}-2\delta_{-4}\\ -\delta_{-2}+4\delta_0-\delta_2\\ -2\delta_4+\delta_6\rangle\end{array} \arrow[l, shift left, "L_6"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_6"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-7}-\delta_{-5}-3\delta_{-3}\\ +3\delta_{-1}-3\delta_1\\ -3\delta_3-\delta_5+\delta_7\rangle\end{array} \arrow[l, shift left, "L_7"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_7"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{l}\langle\delta_{-8}-4\delta_{-4}\\ +6\delta_0-4\delta_4-\delta_8\rangle\end{array} \arrow[l, shift left, "L_8"] \arrow[d, shift left, "G_8"] \arrow[d, leftarrow, shift right, "-D"'] \\
m{=}3 & \begin{array}{r}\langle\delta_{-3}-3\delta_{-1}\\ +3\delta_1-\delta_3\rangle\end{array} \arrow[r, shift left, "M"] \arrow[d, shift left, "G_3"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-4}-2\delta_{-2}\\ +2\delta_2-\delta_4\rangle\end{array} \arrow[l, shift left, "L_4"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_4"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-5}-\delta_{-3}\\ -2\delta_{-1}-2\delta_1\\ +\delta_3-\delta_5\rangle\end{array} \arrow[l, shift left, "L_5"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_5"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-6}-3\delta_{-2}\\ +3\delta_2-\delta_6\rangle\end{array} \arrow[l, shift left, "L_6"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_6"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-7}-\delta_{-5}-3\delta_{-3}\\ -3\delta_{-1}+3\delta_1\\ +3\delta_3-\delta_5-\delta_7\rangle\end{array} \arrow[l, shift left, "L_6"] \arrow[d, shift left, "G_6"] \arrow[d, leftarrow, shift right, "-D"'] \\
m{=}2 & \begin{array}{r}\langle\delta_{-2}-2\delta_0\\ +\delta_2\rangle\end{array} \arrow[r, shift left, "M"] \arrow[d, shift left, "G_2"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-3}-\delta_{-1}\\ -\delta_1+\delta_3\rangle\end{array} \arrow[l, shift left, "L_3"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_3"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-4}-2\delta_0\\ +\delta_4\rangle\end{array} \arrow[l, shift left, "L_4"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_4"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-5}+\delta_{-3}\\ -2\delta_{-1}-2\delta_1\\ +\delta_3+\delta_5\rangle\end{array} \arrow[l, shift left, "L_5"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_5"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-6}+2\delta_{-4}\\ -\delta_{-2}-4\delta_0-\delta_2\\ +2\delta_4+\delta_6\rangle\end{array} \arrow[l, shift left, "L_6"] \arrow[d, shift left, "G_6"] \arrow[d, leftarrow, shift right, "-D"'] \\
m{=}1 & \langle\delta_{-1}-\delta_1\rangle \arrow[r, shift left, "M"] \arrow[d, shift left, "G_1"] \arrow[d, leftarrow, shift right, "-D"'] & \langle\delta_{-2}-\delta_2\rangle \arrow[l, shift left, "L_2"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_2"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-3}+\delta_{-1}\\ -\delta_1-\delta_3\rangle\end{array} \arrow[l, shift left, "L_3"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_3"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-4}+2\delta_{-2}\\ -2\delta_2-\delta_4\rangle\end{array} \arrow[l, shift left, "L_4"] \arrow[r, shift left, "M"] \arrow[d, shift left, "G_4"] \arrow[d, leftarrow, shift right, "-D"'] & \begin{array}{r}\langle\delta_{-5}+3\delta_{-3}\\ +2\delta_{-1}-2\delta_{-1}\\ -3\delta_{-3}-\delta_{-5}\rangle\end{array} \arrow[l, shift left, "L_5"] \arrow[d, shift left, "G_5"] \arrow[d, leftarrow, shift right, "-D"'] \\
m{=}0 & \langle\delta_0\rangle \arrow[r, shift left, "M"] & \langle\delta_{-1}+\delta_1\rangle \arrow[l, shift left, "L_1"] \arrow[r, shift left, "M"] & \begin{array}{r}\langle\delta_{-2}+2\delta_0\\ +\delta_2\rangle\end{array} \arrow[l, shift left, "L_2"] \arrow[r, shift left, "M"] & \begin{array}{r}\langle\delta_{-3}+3\delta_{-1}\\ +3\delta_1+\delta_3\rangle\end{array} \arrow[l, shift left, "L_3"] \arrow[r, shift left, "M"] & \begin{array}{l}\langle\delta_{-4}+4\delta_{-2}\\ +6\delta_0+4\delta_2+\delta_4\rangle\end{array} \arrow[l, shift left, "L_4"] \\
 & n{=}0 & n{=}1 & n{=}2 & n{=}3 & n{=}4
\end{tikzcd}
$$

There are some interesting identities for operators appearing in this subrepresentation (W_{nm}) of the binomial grid.

Theorem 9 (Binomials) *Going two steps in negative direction within the horizontal ladders* $m = 0, 1$ *or vertical ladders* $n = 0, 1$*, respectively, means*

multiplication by some quadratic polynomial. With (36) we get

$$\begin{array}{lcll} L_{n-1}L_n & = & n^2 - X^2 & : \; W_{n,0} \to W_{n-2,0} \\ L_n L_{n+1} & = & (n+1)^2 - X^2 & : \; W_{n,1} \to W_{n-2,1} \\ G_{m-1}G_m & = & X^2 - m^2 & : \; W_{0,m} \to W_{0,m-2} \\ G_m G_{m+1} & = & X^2 - (m+1)^2 & : \; W_{1,m} \to W_{1,m-2} \end{array}$$

Proof The relations (43) and (33), (34) allow to show by induction that

$$\begin{array}{rcll} W_{n,0} & \subseteq & \ker G_n & \text{for all } n \in \mathbb{N}_0 \\ W_{0,m} & \subseteq & \ker L_m & \text{for all } m \in \mathbb{N}_0. \end{array}$$

Then the statement of the theorem follows directly with the relation (36):

$$\begin{array}{llll} L_{n-1}L_n & = L_{n-1}L_n - G_{n-1}G_n & = n^2 - X^2 & : W_{n,0} \to W_{n-2,0} \\ L_n L_{n+1} & = L_n L_{n+1} - G_n G_{n+1} & = (n+1)^2 - X^2 & : W_{n,1} \to W_{n-2,1} \\ G_{m-1}G_m & = G_{m-1}G_m - L_{m-1}L_m & = X^2 - m^2 & : W_{0,m} \to W_{0,m-2} \\ G_m G_{m+1} & = G_m G_{m+1} - L_m L_{m+1} & = X^2 - (m+1)^2 & : W_{1,m} \to W_{1,m-2} \end{array}$$

♦

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